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POSTULATES FOR ASSERTION, CONJUNCTION, NEGATION, AND EQUALITY

By Edward V. Huntington

POSTULATES FOR ASSERTION, CONJUNCTION, NEGATION, AND EQUALITY[1]

By Edward V. Huntington

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Introduction

The purpose of this paper is to present an "abstract deductive theory" so constructed that one of its "concrete interpretations" shall be a generalized form of the calculus of propositions in ordinary logic.

In the last analysis, every "abstract deductive theory" is a sequence of statements written down in a book by some human agent—each such statement being expressed in terms of some predetermined "base" or list of primitive symbols, and the steps by which later statements are derived from earlier ones being authorized by some pre-assigned list of "postulates."

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[1] Presented to the American Mathematical Society and the Association for Symbolic Logic at a joint session held in Cambridge, Massachusetts, September 1, 1936. An outline of part of this paper was printed (under the title: Mathematical postulates for the logical operations of assertion, conjunction, negation, and equality) in the Proceedings of the National Academy of Sciences, vol. 22, pp. 291–296, May, 1936. (On p. 295, the theorem there numbered 51 should read: $a^* \dashv Z^* = Z^*$.)

The present theory may be regarded as a clarification and extension of C. I. Lewis's theory of "strict implication" (Lewis and Langford, Symbolic Logic, 1932). Professor Lewis has authorized and requested me to say that he accepts this presentation as a correct account of the formal properties of his system.

The use of the "subclass T" (corresponding roughly to the Frege assertion sign) was first suggested by the present writer in 1933 (Transactions Amer. Math. Soc., vol. 35, p. 291, 1933; see also Bull. Amer. Math. Soc., vol. 40, p. 127, 1934, and Fundamenta Mathematicae, vol. 25, pp. 147–156, 1935).

As to the equality sign, postulates for the symbol $=$ regarded as a dyadic relation were given in my paper on "Effective equality and effective implication in formal logic," (Proc. Nat. Acad. Sci., vol. 21, pp. 266–271, 1935). In the present paper, the postulates for equality are given in terms of a new primitive symbol $\equiv$ (read "quad"), which is a binary operator. The relation $a = b$ is defined as meaning $a \equiv b$ in T, so that the properties of the dyadic relation $a = b$ are thrown back upon the postulated properties of the binary operator $\equiv$ and the subclass T. The relation $a = b$ thus becomes interpretable in the general sense of "interchangeability" (for any specified purpose) and not merely in the restricted sense of "mutual implication." Further, the

The "base" of the abstract deductive theory here considered is $(K, T, \times, ', \equiv)$, where K is a class of elements $a, b, c, \ldots$; T is a subclass in K; $\times$ (read "times") and $\equiv$ (read "quad") are binary operators in K; and $'$ (read "dash") is a unary operator in K. (For brevity, ab will be written for $a \times b$).

The "postulates" of the theory are listed as Postulates 1–17, below. It is understood that the letters $a, b, c, \ldots$ are variables, each of which may stand for any element of K; that is, in any postulate or theorem, a letter such as a may be "replaced throughout" by any expression which stands for an element of K. ("Principle of substitution.")

The notation $X \rightarrow Y$ (read "X leads to Y") means: "If, in the orderly development of the theory, we find the statement X already

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circularity in the use of the sign " = " in Lewis's definition 11.03 is avoided; and the principle of "replacement of equals by equals," which is postulated as a rule of procedure by Lewis and subsequent writers (including B. A. Bernstein, E. J. Nelson, and others) is here deduced as a theorem. The symbol " = Df " is not required. The concept "$a \equiv b$ in T" is similar to, though not quite the same as, the logogram "$|a = b|$" introduced by A. A. Bennett and C. A. Baylis in their paper on "A calculus for propositional concepts" (Mind, vol. 44, pp. 152–167, 1935).

The postulates for Boolean algebra are adapted from my "fourth set" of 1933 (Trans. Amer. Math. Soc., vol. 35, pp. 280–284, with corrections on p. 557 and p. 971; see also Mind, vol. 42, pp. 203–207, 1933).

As to the central theorems 44 and 74, Th. 44 was first proved from Lewis's postulates in 1934 (Bull. Amer. Math. Soc., vol. 40, pp. 729–735, 1934). The fact that 74 follows from this proof as a corollary was communicated to me by Professor Lewis in October, 1934, and a simple proof of this fact was later given to me by W. T. Parry. [An independent proof of 74 from Lewis's postulates is given by Tang Tsao-Chen (Bull. Amer. Math. Soc., vol. 42, pp. 743–746, 1936).] The proof of 74 from our present postulates is much simpler than the earlier proofs, and requires no reference to Lewis's postulates.

The relation here called "Boolean implication," $ab = a$, is the same as the relation called "effective implication" in my paper of 1935.

As to the examples for consistency and independence, the earliest example of a Lewis system with eight elements is due to W. T. Parry (Mind, vol. 43, pp. 78–80, 1934). Most of the other examples (including one with sixteen elements, and two that prove the independence of Lewis's Postulate B8) are new.

The last section presents a new direct comparison between the systems of "strict implication" and "material implication."

For the convenience of readers who may not have ready access to the earlier literature, all the proofs are given in full, so that the paper is entirely self-contained.

established, then we are authorized to write down the statement Y". Thus the arrow, $\rightarrow$, is merely an abbreviation of language, and not one of the primitive concepts of the system. Obviously, $X \rightarrow X$. Also, if $X \rightarrow Y$ and $Y \rightarrow Z$, then $X \rightarrow Z$. To indicate that both $X \rightarrow Y$ and $Y \rightarrow X$, we write $X \rightleftarrows Y$. (It should be noted that the arrow, $\rightarrow$, is used only between statements, not between elements of K.) Further: The notation "$X \& Y$" means simply "X and Y"; and the notation "$\exists a$ (. . .)" means "there exists at least one element a such that (. . .)."

The theorems of the system are divided into two groups—those that are deducible from Postulates 1–14, and those that require for their proof one or more of the Postulates 15–17. Again, theorems (or postulates) which contain the arrow, $\rightarrow$, may be called "rules of procedure," as distinguished from "formal laws." (Since at least one rule of procedure is indispensable in any abstract deductive system, it appears to be largely a matter of taste how many such rules are admitted, either among the postulates or among the theorems.)

The "consistency" of the postulates is established by exhibiting finite examples of systems $(K, T, \times, ', \equiv)$ which satisfy all the postulates. The (serial) "independence" of the postulates is established in a similar way, by exhibiting finite examples of systems $(K, T, \times, ', \equiv)$ which satisfy some but not all of the postulates.

The "concrete interpretation" which is of chief interest is the system $(K, T, \times, ', \equiv)$ exhibited in the example described as a generalized "calculus of propositions." But the question whether this (non-finite) example really does possess all the properties demanded by the postulates is a question of fact requiring a critical examination of the actual usages of language; and logicians may not all agree as to the linguistic facts. It is a virtue of the abstract formulation of the theory that it makes these debatable linguistic questions precise and definite, even though it does not pretend to answer them completely.

§1. The Postulates.

Base: $(K, T, \times, ', \equiv)$. *Language:* "$\rightarrow$", "$\&$", "$\exists$".

Postulate 1. a in K. $\&$. b in $K :\rightarrow: a \times b$ in K. (Read: a times b.)
Postulate 2. a in $K .\rightarrow. a'$ in K. (Read: a dash.)
Postulate 3. a in $K . \&. b$ in $K :\rightarrow: a \equiv b$ in K. (Read: a quad b.)
Postulate 4 (a). a in $T .\rightarrow. a$ in K. (T a subclass in K.)
(b). $\exists a$: (a in T). (T non-empty.)
(c). $\exists a$: (a in $K . \&.$ a not in T). (T non-exhaustive.)

These four postulates, 1–4, are "descriptive postulates," or "postulates of closure." They are all independent of each other. [But after any one of Postulates 11–14 is included, Postulate 4b may be replaced by the weaker form: $\exists a: (a \text{ in } K)$; and Postulate 4c may be omitted altogether without seriously affecting the theory, as explained under Theorem 54c.] The following postulates are understood to be required only in cases in which Postulates 1–4 are valid. (As usual, ab is written for $a \times b$.)

POSTULATE 5. a in T .&. b in $T :\longrightarrow: ab$ in T. (Rule of adjunction.)
POSTULATE 6. ab in $T :\longrightarrow: a$ in T. (Rule of reduction.)
DEFINITION A. $a = b :\rightleftarrows : a \equiv b$ in T. (Read: a equal to b.)

It should be noted that the symbol "$=$" introduced by Definition A is part of the abstract system and has here no properties except such as are deducible from the postulates governing the symbols $\equiv$ and T. The notation "$a = b$" is merely a convenient abbreviation for the statement "$a \equiv b$ in T".

POSTULATE 7. a in T .&. $a = b :\longrightarrow: b$ in T. (Rule of equality for T.)
POSTULATE 8. $a = b :\longrightarrow: ac = bc$. (Rule of equality for $\times$.)
POSTULATE 9. $a = b :\longrightarrow: a' = b'$. (Rule of equality for $'$.)
POSTULATE 10. $a = b :\longrightarrow: (a \equiv c) = (b \equiv c)$. (Rule of equality for $\equiv$.)

These ten postulates are "procedural postulates," since they involve the arrow, $\longrightarrow$. All the remaining postulates are "formal laws."[2]

POSTULATE 11. $(a \equiv b) = (b \equiv a)$. (Commutative law for $\equiv$.)
POSTULATE 12. $ab = ba$. (Commutative law for $\times$.)
POSTULATE 13. $(ab)c = a(bc)$. (Associative law for $\times$.)
POSTULATE 14. $(a'b')'\,(a'b')' = a$. (Law of expansion.)

These postulates 5–14 are independent of each other, and of 1–4. *The fourteen postulates 1–14 form the basis of Part I of our theory.* In view of Theorem 33: $aa' = bb'$, we are justified in introducing the following definition:

DEFINITION B. $Z = aa'$. (Read: the "zero element" of the system.)

[2] It should be noted, however, that each of these formal laws is supposed to be preceded by the hypothesis that the letters involved are elements of K. For example, Postulate 12, when written out fully, should read:

$$a \text{ in } K \text{ .\&. } b \text{ in } K :\longrightarrow: ab \equiv ba \text{ in } T.$$

In a sense, therefore, *all* the postulates, including the "formal laws," may be regarded as "rules of procedure," since they all involve the arrow.

The remaining postulates (Part II) are understood to be required only in cases in which Postulates 1–14 are valid.

DEFINITION C. $a^* = (a \equiv Z)$.
(Read: a star. Lewis's curl-diamond a.)

DEFINITION D. $(a \prec b) = (ab')^*$.
(Read: a hook b. Lewis's "strict implication.")

POSTULATE 15. $(a \prec b)(b \prec a) = (a \equiv b)$.
(Law of equivalence. "Mutual hook.")

POSTULATE 16. $[(a \prec b)(b \prec c) \prec (a \prec c)]$ in T.
(Law of the syllogism.)

POSTULATE 17. $[a(a \prec b) \prec b]$ in T. (Law of inference.)

These postulates 15–17 are independent of each other and of 1–14.

§1a. A Possible Interpretation: A Generalized "Calculus of Propositions"

Before proceeding with the deduction of theorems, we mention at once the "concrete interpretation" which we have chiefly in mind. In this concrete system, the abstract symbols (K, T, $\times$, $'$, $\equiv$) are interpreted as follows:

K is the class of "propositions" (better described as "sentences").

T is the subclass of propositions which are "asserted," so that the statement "a is in T" means: "a is asserted."

ab is the proposition known as the "logical product" (or "conjunction") of a and b, so that the statement "ab is in T" means: "the joint proposition, a and b, is asserted."

a' is the proposition known as the "contradictory" of a, so that the statement "a' is in T" means: "the contradictory of a is asserted."

$a \equiv b$ is the proposition known as the "interchangeability" of a and b for some specified purpose, so that the statement "$a \equiv b$ is in T" means: "a is interchangeable with b for the purpose in hand."

Under this interpretation of the basic symbols (K, T, $\times$, $'$, $\equiv$), the derived symbols introduced by Definitions B, C, D may be interpreted as follows:

Z is "absurdity," that is, Z is the conjunction of any proposition with its contradictory. If Z were asserted, then *every* proposition would be asserted, which would be absurd.

a^* is the proposition known as the "absurdity of a," so that the statement "a^* is in T," or "$a \equiv Z$ is in T," means: "the absurdity of a is asserted," or "a is absurd."

$a \dashv b$ is the proposition known as the "implication of a to b" so that the statement: "$a \dashv b$ is in T" means: "a does imply b."

As already intimated, the question whether this concrete system $(K, T, \times, ', \equiv)$ really does satisfy all the postulates is a question which belongs in the field of linguistics rather than in the field of mathematics, and will not be discussed in the present paper.

An alternative example is obtained by replacing the word "asserted" by the word "demonstrable."

Much simpler examples, containing only a finite number of elements, are given below (in order to establish the consistency and independence of the postulates). The reader may find it convenient to refer to Examples 0.1 and 0.6 at this point.

§2. Theorems on the Equality Sign

Part I, depending only on Postulates 1–14, comprises Theorems 21–69. The theorems 21–29 develop the properties of the equality sign, $=$. In the proofs, the obvious properties of the arrow notation, and the use of Definition A, will be understood without special mention. (All the proofs are so arranged that the word "hence" refers to statements not more than two lines back.)

21. ab in $T.\rightarrow.ba$ in T.

Proof. By 7, ab in $T.\&.ab \equiv ba$ in $T:\rightarrow:ba$ in T. But by 12, $ab \equiv ba$ is in T. Hence ab in $T.\rightarrow.ba$ in T.

22. $a \equiv b$ in $T.\rightarrow.b \equiv a$ in T.

Proof similar, 7, 11 being used in place of 7, 12.

23. ab in $T.\rightarrow.b$ in T.

Proof. By 21, ab in $T.\rightarrow.ba$ in T. By 6, ba in $T.\rightarrow.b$ in T.
Hence, ab in $T.\rightarrow.b$ in T.

24. $a = b.\rightarrow.ca = cb$.

Proof. By 12 (and A), $ac \equiv ca$ is in T.
By 10, $ac \equiv ca$ in $T.\rightarrow.(ac \equiv d) \equiv (ca \equiv d)$ in T.
Hence, $(ac \equiv d) \equiv (ca \equiv d)$ is in T.
By 7, $ac \equiv d$ in $T.\&.(ac \equiv d) \equiv (ca \equiv d)$ in $T:\rightarrow:ca \equiv d$ in T.
Hence, $ac \equiv d$ in $T.\rightarrow.ca \equiv d$ in T.
By 22, $ca \equiv d$ in $T.\rightarrow.d \equiv ca$ in T.
Hence, $ac \equiv d$ in $T.\rightarrow.d \equiv ca$ in T. (a)
By 8, $a \equiv b$ in $T.\rightarrow.ac \equiv bc$ in T.
By (a), $ac \equiv bc$ in $T.\rightarrow.bc \equiv ca$ in T.

Hence, $a \equiv b$ in $T. \rightarrow .bc \equiv ca$ in T.
By (a), $bc \equiv ca$ in $T. \rightarrow .ca \equiv cb$ in T.
Hence, $a \equiv b$ in $T. \rightarrow .\ ca \equiv cb$ in T. That is, $a = b. \rightarrow .ca = cb$.

25. $a = b. \rightarrow .(c \equiv a) = (c \equiv b)$.

Proof. As in the proof of 24, using 11 in place of 12, we find
$(a \equiv c) \equiv d$ in $T. \rightarrow .d \equiv (c \equiv a)$ in T. (b)
Hence, using 10 and (b) in place of 8 and (a), we find
$a \equiv b$ in $T. \rightarrow .(c \equiv a) \equiv (c \equiv b)$ in T.

26. $a = a$. (Law of identity.)

Proof. By 10, $(a'b')'(a'b')' \equiv a$ in $T. \rightarrow .[(a'b')'(a'b')' \equiv a] \equiv [a \equiv a]$ in T.
But by 14, $(a'b')'(a'b')' \equiv a$ is in T.
Hence, $[(a'b')'(a'b')' \equiv a] \equiv [a \equiv a]$ is in T.
Hence by 7, $a \equiv a$ is in T. That is, $a = a$.

27. $a = b. \rightarrow .b = a$.

Proof by 22 and A.

28. $a = b.\,\&.b = c : \rightarrow : a = c$.

Proof. By 10, $a \equiv b$ in $T. \rightarrow .\ (a \equiv c) \equiv (b \equiv c)$ in T.
By 22, $(a \equiv c) \equiv (b \equiv c)$in $T. \rightarrow .(b \equiv c) \equiv (a \equiv c)$ in T.
Hence, $a \equiv b$ in $T. \rightarrow .\ (b \equiv c) \equiv (a \equiv c)$ in T. (a)
From (a), $a \equiv b$ in $T.\,\&.b \equiv c$ in $T : \rightarrow : b \equiv c$ in $T.\,\&.(b \equiv c) \equiv (a \equiv c)$ in T.
By 7, $b \equiv c$ in $T.\,\&.(b \equiv c) \equiv (a \equiv c)$ in $T : \rightarrow : a \equiv c$ in T.
Hence, $a \equiv b$ in $T.\,\&.b \equiv c$ in $T : \rightarrow : a \equiv c$ in T.
That is, by A, $a = b.\,\&.b = c : \rightarrow : a = c$.

28a. $a = b.\,\&.c = d : \rightarrow : ac = bd$.

Proof. By 8, 24, $a = b.\,\&.c = d : \rightarrow : ac = bc.\,\&.bc = bd$.
By 28, $ac = bc.\,\&.bc = bd : \rightarrow : ac = bd$.
Hence, $a = b.\,\&.c = d : \rightarrow : ac = bd$.

28b. $a = b.\,\&.c = d : \rightarrow : (a \equiv c) = (b \equiv d)$.

Proof by 10, 25, 28.

In similar manner, every theorem authorizing the replacement of "equals by equals" which is required in this paper can be proved by direct deduction from the postulates. In order to avoid the labor of writing out the direct proof of each such theorem separately as needed, the following general theorem of replacement is here inserted. In establishing this general theorem, the indirect method of proof (reductio ad absurdum) is freely employed; but it should be clearly

noted that the indirect method of proof plays no essential part in the paper. The purpose of the general theorem is merely to convince the reader that each particular theorem can be proved directly if desired.

29. *Theorem of replacement.* Let ξ be any expression which stands for an element of K (that is, any expression built up in accordance with Postulates 1, 2, 3, out of parentheses, the letters $a, b, c, \ldots$, and the signs $\times, ', \equiv$); and suppose that at some point in the expression ξ there occurs the letter a. Let η be another expression which differs from ξ only in having this particular a replaced by b. Then if we find $a = b$ established, we are authorized to write down $\xi = \eta$. That is, when ξ and η have the meanings indicated, we have the theorem: $a = b.\rightarrow.\xi = \eta$.

Proof.[3] Let n be the number of marks (that is, parentheses; letters $a, b, c, \ldots$; or signs $\times, ', \equiv$) in the expression ξ. Then there are five possible cases to consider.

Case 1. ξ may have the form of a product, $\xi_1 m$, where ξ_1 is an expression involving the a in question and containing fewer than n marks, while m is an expression not involving the a. In this case η will have the form $\eta_1 m$, where η_1 is the same as ξ_1 with the a replaced by b, while m is unchanged. In this case we can prove that *if* the statement $a \equiv b$ in $T.\rightarrow.\xi \equiv \eta$ in T *were false, then* the statement $a \equiv b$ in $T.\rightarrow.\xi_1 \equiv \eta_1$ in T *would also be false.*

For, if this latter statement were true, then by 8 we would have $a \equiv b$ in $T.\rightarrow.\xi_1 m \equiv \eta_1 m$ in T; that is, $a \equiv b$ in $T.\rightarrow.\xi \equiv \eta$ in T; contrary to the supposition.

Case 2. ξ may have the form $m\xi_1$. In this case we can prove the same result, by using 24 instead of 8.

Case 3. ξ may have the form $\xi_2 \equiv m$, where ξ_2 involves the a in question and contains fewer than n marks, while m does not involve the a. In this case η will have the form $\eta_2 \equiv m$. In this case we can prove that *if* the statement $a \equiv b$ in $T.\rightarrow.\xi \equiv \eta$ in T *were false, then* the statement $a \equiv b$ in $T.\rightarrow.\xi_2 \equiv \eta_2$ in T *would also be false.*

For, if this latter statement were true, then by 10 we would have $a \equiv b$ in $T.\rightarrow.(\xi_2 \equiv m) \equiv (\eta_2 \equiv m)$ in T; that is, $a \equiv b$ in $T.\rightarrow.\xi \equiv \eta$ in T; contrary to the supposition.

Case 4. ξ may have the form $m \equiv \xi_2$. In this case we can prove the same result, by using 25 instead of 10.

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[3] For a valuable suggestion in regard to this proof, I am indebted to Dr. J. B. Rosser.

Case 5. ξ may have the form $(\xi_3)'$, where ξ_3 involves the a in question and contains fewer than n marks. In this case η will have the form $(\eta_3)'$, where η_3 differs from ξ_3 only in having the a replaced by b. In this case we can prove that *if* the statement $a \equiv b$ in $T . \rightarrow . \xi \equiv \eta$ in T *were false, then* the statement $a \equiv b$ in $T . \rightarrow . \xi_3 \equiv \eta_3$ in T *would also be false.*

For, if this latter statement were true, then by 9 we would have $a \equiv b$ in $T . \rightarrow . (\xi_3)' \equiv (\eta_3)'$ in T; that is, $a \equiv b$ in $T . \rightarrow . \xi \equiv \eta$ in T; contrary to the supposition.

By successive applications of these five results, we can prove that *if* the statement $a \equiv b$ in $T . \rightarrow . \xi \equiv \eta$ in T *were false, then* the statement $a \equiv b$ in $T . \rightarrow . \xi_k \equiv \eta_k$ in T (where ξ_k has been reduced to the single element a, and η_k to the single element b) *would also be false.* But this final statement, $a \equiv b$ in $T . \rightarrow . a \equiv b$ in T, is obviously true. Hence the supposition must have been false, and thus the theorem 29 is established.

Note. These theorems 26, 27, 28, 29 show that within any system $(K, T, \times, ', \equiv)$ *which obeys Postulates 1–14, the sign "* $=$ *" introduced by Definition A may be handled exactly like the ordinary equality sign in mathematics.* In what follows, free use will be made of this fact, without special mention on each occasion.

§3. Boolean Algebra Properties

The five postulates 1, 2, 12, 13, 14, together with theorems 26–29, make the system a Boolean algebra with respect to $(K, \times, ')$. Hence the following theorems, 31–39, may be classified as among the "Boolean algebra properties" of the system.

31. $a'' = a$. (Law of double negation.)

Proof. By 14, 28a, $aa' = [(a'a'')'(a'a''')'][(a''a'')'(a''a''')']$.
By 14, 28a, $a'a'' = [(a''a')'(a''a'')'][(a''' a')'(a''' a'')']$.
Hence by 12, 13, $aa' = a'a''$. Hence, $a'a'' = a''a'''$.
By 14, $a = (a'a'')'(a'a''')'$. Hence, $a = (a''a''')'(a'a''')'$.
By 14, $a'' = (a''' a')'(a''' a'')'$. Hence by 12, $a = a''$.

32. $(ab)'(ab')' = a'$. (By 14, 31.)

33. $aa' = bb'$.

Proof. By 14, 32, $aa' = [(a'b')'(a'b'')'][(ab')'(ab'')']$.
By 14, $bb' = [(b'a)'(b'a')'][(b''a)'(b''a')']$.
Hence by 12, 13, $aa' = bb'$.

34. Definition. $Z = aa'$. (The "zero element" of the system.)

This definition is justified by theorem 33, which shows that aa' is a unique element in the system.

35. $aZ' = a$. (Z' is called the "universe element" of the system.)

Proof. By 32, 34, $Z' = (ZZ)'(ZZ')' = (ZZ)'(Z)'$. (a)
By 34, (a), 13, 12, 34, $Z = ZZ' = Z(ZZ)'Z' = (ZZ')(ZZ)' = (Z)(ZZ)'$,
whence by 13, 34, $ZZ = Z[Z(ZZ)'] = (ZZ)(ZZ)' = Z$. (b)
By (a), (b), $Z' = Z'Z'$. (c)
By 14, 34, 12, 13, (c), 34, 14,
$aZ' = [(a'a)'(a'a')']Z' = Z'(a'a')'Z'$
$= (Z'Z')(a'a')' = Z'(a'a')' = (a'a)'(a'a')' = a$.

36. $aa = a$. (Idempotent law for $\times$.)

Proof. By 35, 34, 32, $(aa)' = (aa)'Z' = (aa)'(aa')' = a'$.
Hence by 31, $aa = a$.

37. $aZ = Z$.

By 34, 13, 36, 34, $aZ = a(aa') = (aa)a' = aa' = Z$.

38. $a(a'b')' = a$.

Proof. By 14, 13, 36, 14,
$a(a'b')' = (a'b)'(a'b')'(a'b')' = (a'b)'(a'b')' = a$.

39. $a(b'c')' = [(ab)'(ac)']'$. (Distributive law for $\times$.)[4]

Proof.[5] By 32, 13, $[a(b'c')']' = [a(b'c')'b]'\,[a(b'c')'b']'$. (a)
By 12, 13, 38, $a(b'c')'b = a[b(b'c')'] = ab$. (b)
By 32, 12, 13, 14,

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[4] If we introduce the following familiar definition:

DEFINITION E. $a \vee b \;.=.\; (a'b')'$ (read: a wedge b); then we have at once the following five theorems, which make the system a Boolean algebra with respect to $(K, \vee, ')$:

(1) a in K .&. b in K :⟶: $a \vee b$ in K.
(2) a in K .⟶. a' in K.
(3) $(a \vee b) = (b \vee a)$. (Commutative law for $\vee$.)
(4) $(a \vee b) \vee c \;.=.\; a \vee (b \vee c)$. (Associative law for $\vee$.)
(5) $(a' \vee b')' \vee (a' \vee b)' = a$. (Law of expansion for $\vee$.)

From 38 and 39 we then have:

$a(a \vee b) = a$.
$a(b \vee c) = ab \vee ac$. (Distributive law for $\times$.)
$a \vee bc = (a \vee b)(a \vee c)$. (Distributive law for $\vee$.)

[5] For a marked simplification of my 1933 proof of this theorem 39, I am indebted to Professor A. A. Bennett.

$a(b'c')'b' = a(b'c')'[(bc)'(bc')'] = a[(c'b)'(c'b')'](bc)' = ac(bc)'.$ (c)
By (b), (c) in (a), $[a(b'c')']' = (ab)'[ac(bc)']'.$ (d)
From (d), $[a(c'b')']' = (ac)'[ab(cb)']',$
whence by 12, $[a(b'c')']' = (ac)'[ab(bc)']'.$ (e)
By 36, (d), (e), 12, 13, 31, 38, 38,

$$\begin{aligned}[a(b'c')']' &= [a(b'c')']'[a(b'c')']' = \\ &= (ab)'[ac(bc)']'(ac)'[ab(bc)']' = \\ &= (ab)'[(ab)''(bc)']'(ac)'[(ac)''(bc)']' = \\ &= (ab)'(ac)'.\end{aligned}$$

Hence by 31, $a(b'c')' = [(ab)'(ac)']'.$

§4. "Boolean" Implication

The following theorems, 41–49, are also "Boolean algebra properties" of the system.

41. $ab' = Z.\rightleftarrows .ab = a.$

Proof. By 39, 34, 35,
$b'(a''b')' = [(b'a')'(b'b)']' = [(b'a')'Z']' = [(b'a')']',$
whence by 31, 12, $b'(ab')' = a'b'.$ (a)
By 31, 35, $ab = a[b']' = a[b'(Z)']'.$ (b)
By 38, (a), $a = a[a'b']' = a[b'(ab')']'.$ (c)
From (b), (c), $ab' = Z.\rightarrow .ab = a,$ which is the first part of the theorem.
By 38, 31, 12, 13, $ab' = a[b'(b''a)'] = b'[a(ab)'].$ (d)
By 37, $Z = b'Z = b'[a(a)'].$ (e)
From (d), (e), $ab = a.\rightarrow .ab' = Z,$ which is the second part.

42. $a'b' = b'.\rightleftarrows .ab = a.$

Proof. By 31, $ab = a(b')'.$ By 38, $a = a(a'b')'.$
Hence, $a'b' = b'.\rightarrow .ab = a,$ which is the first part.
By 38, 31, 12, $b' = b'(b''a)' = b'(ab)'.$ By 12, $a'b' = b'(a)'.$
Hence, $ab = a.\rightarrow .a'b' = b',$ which is the second part.

43. Definition. $a \leqslant b.\rightleftarrows .ab' = Z.$ (Read: a implies b.)

44. $a \leqslant b.\rightleftarrows .ab = a.$ (By 43, 41.)

This relation, $a \leqslant b,$ has the fundamental properties commonly associated with the word "implication," as follows:

45. $a \leqslant a.$ (Self-implication.)

Proof by 44, 36, 26.

46. $a \leqslant b.\rightleftarrows .b' \leqslant a'.$ (Inverse implication.)

Proof by 43, 31, 12.

47. $a \leqslant b.\&.b \leqslant a :\rightleftarrows: a = b$. (Mutual implication, or equivalence.)

Proof by 44, 12.

48. $a \leqslant b.\&.b \leqslant c :\rightarrow: a \leqslant c$. (Syllogism.)

Proof. By 44, 43, 28a, 13, 36, 37, 44, 43,
$a \leqslant b.\&.b \leqslant c :\rightarrow: ab = a.\&.bc' = Z :\rightarrow: abbc' = aZ :\rightarrow: (ab)c' = Z :\rightarrow: (a)c' = Z :\rightarrow: a \leqslant c$.

49. a in $T.\&.a \leqslant b :\rightarrow: b$ in T. (Inference.)

Proof. By 44, 27, 7, 23,
a in $T.\&.a \leqslant b :\rightarrow: a$ in $T.\&.a = ab :\rightarrow: ab$ in $T :\rightarrow: b$ in T.

The following theorem will be useful in the proof of 59.

49a. $ab = Z.\&.a'b' = Z :\rightarrow: b = a'$.

Proof. By 41, 31, $ab = Z.\rightarrow.ab' = a$;
and by 41, 12, $a'b' = Z.\rightarrow.ab' = b'$. Hence the theorem.

§5. Asserted and Rejected Elements. The Subclass T'

The theorems of this section (51–59), like all the preceding theorems, depend only on Postulates 1–14. (Note that 52c and 54c, together with Postulate 4c, may all be omitted, if preferred, in view of 55.)

51. Z' is in T.

Proof. By 7, a in $T.\&.a \equiv aZ'$ in $T :\rightarrow: aZ'$ in T.
But by 35, 27, $a \equiv aZ'$ is in T. Hence a in $T.\rightarrow.aZ'$ in T.
By 23, aZ' in $T.\rightarrow.Z'$ in T. Hence, a in $T.\rightarrow.Z'$ in T.
That is, if there is any element in T, then Z' is in T. But by 4b, there is at least one element a such that a is in T. Hence Z' is in T.

52. Z in $T.\rightarrow.b$ in T; whence

52c. Z is not in T.

Proof. By 34, Z in $T.\rightarrow.bb'$ in T. By 6, bb' in $T.\rightarrow.b$ in T. Hence, Z in $T.\rightarrow.b$ in T.
That is, if Z is in T, then every element of K is in T. But by 4c, there exists at least one element of K which is not in T. Hence, Z is not in T.

The significance of these two theorems will be made clearer if we introduce the following definition of a new subclass, T'.

53. Definition. a in $T' .\rightleftarrows. a'$ in T. (Read: a in T dash.)

The statement: a in T' must not be confused with the statement: a not in T.

If we think of T as the class of "asserted" elements, and of a' as the element "contradictory" to a, then we may conveniently speak of T' as the class of "rejected" elements; but if this terminology is used, it must be clearly understood that a "rejected" element means precisely what definition 53 says it means, namely: "an element whose contradictory is asserted," and not merely: "an element which itself is not asserted." But there may be many elements of K which are not in T and also not in T'—that is, many elements which are neither "asserted" nor "rejected."

In this connection, it should be especially noted that the concept of "class," as a "completed totality of elements" plays no part in the theory of the present paper. Concerning any given object, we are supposed to be able to tell whether or not it has the property described by the phrase "in K"; and concerning any element of K (that is, any object having the property "in K"), we are supposed to be able to tell whether or not it has the further property described by the phrase "in T"; but while it is convenient to speak of K and T as "classes," they may be more properly regarded as "predicates," since the technical "theory of classes" is not involved at any point.

The following theorem states merely that no element a can have both the properties T and T'.

54. a in $T.\&.a$ in $T' :\rightarrow : b$ in T; whence,

54c. a in $T.\rightarrow.a$ not in T'; and a in $T'.\rightarrow.a$ not in T.

Proof. By 53, 5, 52,
a in $T.\&.a$ in $T' :\rightarrow : a$ in $T.\&.a'$ in $T :\rightarrow : aa'$ in $T :\rightarrow : Z$ in $T :\rightarrow : b$ in T.
That is, if a and a' are both in T, then every element of K is in T. But by 4c, there exists at least one element of K which is not in T. Hence the supposition a in $T.\&.a$ in T' cannot hold. Hence, a in $T .\rightarrow.a$ not in T'; and a in $T'.\rightarrow.a$ not in T.

55. Note. If one prefers to avoid the use of the word "not" in the formal theory, one may omit Postulate 4c and Theorem 52c and 54c, replacing them by the non-formal theorem: Z in $T.\rightarrow.$"$T = K$" (where the notation "$T = K$" means that the classes T and K coincide), together with the convention that the case "$T = K$", while formally admissable, is to be regarded as practically trivial.

56. Z is in T'.

Proof by 51, 53.

57. $Z \leqslant a$.

Proof. By 37, 44, 29, $Z \equiv Z$ in $T.\rightarrow.Za \equiv Z$ in $T.\rightarrow.Za = Z.\rightarrow.Z \leqslant a$.

But by 26, $Z \equiv Z$ is in T. Hence $Z \leqslant a$. That is, the zero element "implies" every element.

Note. This theorem 57: $(b = Z. \rightarrow .b \leqslant a)$ must not be confused with the paradoxical proposition 57a: b in $T'. \rightarrow .b \leqslant a$. This latter proposition ("a rejected element implies every element") cannot be proved from our postulates.

58. $a \leqslant Z'$.

Proof. By 35, 44, 29, $a \equiv a$ in $T. \rightarrow .aZ' \equiv a$ in $T. \rightarrow .aZ' = a. \rightarrow .a \leqslant Z'$. But by 26, $a \equiv a$ is in T. Hence $a \leqslant Z'$. That is, the universe element is "implied by" every element.

Note. This theorem 58: $(b = Z'. \rightarrow .a \leqslant b)$ must not be confused with the paradoxical proposition 58a: b in $T. \rightarrow .a \leqslant b$. This latter proposition ("an asserted element is implied by every element") cannot be proved from our postulates.

The following theorem concerns the number of elements (2^n) in a finite Boolean algebra.

59. (1) Any class K containing 2^n elements $(n = 1, 2, 3, \ldots)$ can be made into a Boolean algebra $(K, \times, ')$ by suitably defining the binary operator "times" and the unary operator "dash"; and (2) in any finite Boolean algebra $(K, \times, ')$ the number of elements is some power of 2. (The degenerate case where K contains only a single element may be included under the name Boolean algebra if desired.[6])

Proof of (1). Following a device due to H. M. Sheffer (1920), let K be represented as the class of all factors of a "Boolean integer," B, where a Boolean integer means any integer which has no repeated factor (other than 1). For example, if $B = 210$, then $K =$

1; 2; 3, 6; 5, 10, 15, 30; 7, 14, 21, 42, 35, 70, 105, 210.

(Since the introduction of each new prime doubles the number of elements, any such class K will clearly contain 2^n elements.) Now let ab = the least common multiple of a and b, and let a' = the quotient of B divided by a. Then it is readily seen that Postulates 1, 2, 12, 13, 14 are all satisfied. (Here $Z = B$ itself, and $a \leqslant b$ means "a contains b as a factor.")

[6] H. M. Stone, Postulates for Boolean algebras and generalized Boolean algebras, Amer. Journal of Math., vol. 57, pp. 703–732, 1935, especially page 705.

Proof of (2).[7] In this proof we use the notation "$a \neq b$" for "$a = b$ does not hold," and introduce the definition of a "prime" element as follows:

DEFINITION. A "prime" element p is any element which occurs not more than three times in the table for ab. That is, if p is prime, then $pb = p :\rightarrow: (b = Z' \text{ or } b = p)$; and if a is not prime, then $\exists b$ $(b \neq Z', b \neq a) : ab = a$. (Compare the table for ab in Example 0.6, below, where the prime elements, other than Z', are 2, 3, 5, 9.)[8]

On the basis of this definition, we prove the following lemmas.

Lemma 1. If p, q are distinct primes, then $p'q' = Z$.

Proof. Let $c' = p'q'$. Then by 38, $pc = p(p'q')' = p$,
whence, by the definition of prime, $c = Z'$ or $c = p$. Similarly, $qc = q$, whence $c = Z'$ or $c = q$. But by hypothesis, $p \neq q$.
Hence $c = Z'$; that is, $p'q' = Z$.

Lemma 2. If $u = pqr \ . \ . \ .$ and $v = st \ . \ . \ .$, then $u'v' = Z$.
Here $p, q, r, \ . \ . \ .$ are distinct primes $\neq Z'$, and $s, t, \ . \ . \ .$ are a second set of distinct primes $\neq Z'$, entirely distinct from the first set; $u =$ the product of one or more factors from the first set; and $v =$ the product of one or more factors from the second set. (Similarly, we may let $w =$ the product of one or more factors from a third set having nothing in common with either of the other two sets.)

Proof. By successive applications of 39 we find
$u'v' = (pqr...)' \ (st...)'$
$= [(p's')'(p't')'...(q's')'(q't')'...]'$, whence
by Lemma 1, $u'v' = [Z'Z'...Z'Z'...]' = [Z']' = Z$.

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[7] This proof will be found easier to follow than the proof given in my paper of 1904 (Trans. Amer. Math. Soc., vol. 5, p. 309). Another proof, more closely connected with the theory of "lattices" (Garrett Birkhoff, H. M. MacNeille, etc.), has been pointed out to me by Saunders MacLane as a corollary of his paper "On the representation of finite lattices," presented to the Amer. Math. Soc. on March 30, 1934.

[8] If p is a prime element, then p' may be called a "basic" element. Thus, in Example 0.6, the basic elements, besides Z, are 15, 14, 12, 8. If K is represented as a finite class of regions in a square, with $ab =$ the region common to the regions a and b, and $a' =$ the region complementary to the region a, then the prime elements are the "large" regions (including the whole square, Z'), while the basic elements are the "small" regions (including the null region, Z); and $a \leqslant b$ means "a within b." The basic elements bear the same relation to the operator $\vee$, that the prime elements bear to the operator $\times$. (The "principle of duality.")

Lemma 3. With the same notation as in Lemma 2 we have

$$u \neq Z', \quad v \neq u, \quad uv \neq u, \quad \text{and} \quad uv \neq uw.$$

Proof. Suppose $u = Z'$, that is, $pqr\ldots = Z'$; then by 35, 37, 13, $p' = p'Z' = p'(pqr\ldots) = Z(qr\ldots) = Z$, whence $p = Z'$, contrary to hypothesis. Again, suppose $v = u$; then by 36 and Lemma 2, $u' = u'v' = Z$, contrary to result just proved. Again, suppose $uv = u$; then by 42 and Lemma 2, $v' = u'v' = Z$, whence $v = Z'$, contrary to result just proved. Again, suppose $uv = uw$; then by 36, 39, and Lemma 2, $(uv)' = (uv)'(uw)' = [(u'u')'(u'w')'(v'u')'(v'w')']' = u'$, whence $uv = u$, contrary to result just proved.

Lemma 4. If $uv = Z$, then $v = u'$. (Same notation as in Lemma 2.)

Proof. By Lemma 2, $u'v' = Z$;
and by 49a, $uv = Z.\&.\ u'v' = Z :\rightarrow : v = u'$.

Lemma 5. In a finite Boolean algebra, if a is not prime, then $a \leqslant p$, where p is a prime $\neq Z'$.

Proof. By 44 and the definition of prime, $\exists\, b(b \neq Z',\ b \neq a)$: $a \leqslant b$. If b is not prime, $\exists\, c(c \neq Z',\ c \neq b)$: $b \leqslant c$, and by 48, 47, $c \neq a$. If c is not prime, $\exists\, d(d \neq Z',\ d \neq c)$: $c \leqslant d$, and by 48, 47, $d \neq b$, $d \neq a$. This process leads to a chain of non-primes, $a \leqslant b \leqslant c \leqslant d \leqslant \ldots$, each element of the chain being distinct from all its predecessors. Hence, since K is finite, we must finally reach an element p which is prime, whence by 47, $a \leqslant p$.

Lemma 6. In a finite Boolean algebra, if a is not prime, then $a = pb$, where p is a prime $\neq Z'$, and $b \neq a$.

Proof. By Lemma 5, $\exists\, p(p \neq Z')$: $a \leqslant p$, whence by 44, 42, $ap = a$ and $a'p' = p'$. Let $b = (a'p)'$. Then by 39, $pb = p(a'p)' = [(pa)'(pp')']' = pa = a$. Also, $b \neq a$. For if $b = a$, then $a = (a'p)'$, whence by 42, $ap' = p'$, whence $p' = p'p' = (ap')(a'p') = aa'pp' = Z$, contrary to the fact that $p \neq Z'$.

Lemma 7. In a finite Boolean algebra, any element a can be expressed as the product of a finite number of distinct primes: $a = Z'pqr\ \ldots$

Proof. If a is not prime, then by Lemma 6, $a = pb$, where $b \neq a$. If b is not prime, then by Lemma 6, $b = qc$, where $c \neq b$, so that $a = pqc$. Also, $c \neq a$. (For, if $c = a$, then $b = qa$ while $a = pb$, whence by 44, $b \leqslant a$ and $a \leqslant b$, whence by 48, $a = b$, contrary to fact.) If c is not prime, then by Lemma 6, $c = rd$, where $d \neq c$, so that $a = pqrd$. Also, $d \neq b$ and $d \neq a$. (For, if $d = b$, then $c = rb$ while $b = qc$, whence $c \leqslant b$

and $b \leqslant c$, whence $b = c$, contrary to fact; and if $d = a$, then $c = ra$, while $a = pqc$, whence $c \leqslant a$ and $a \leqslant c$, whence $a = c$, contrary to fact.) This process leads to a sequence of distinct primes, $a = pqr\ldots$ which must terminate, since K is finite.

Lemma 8. If a finite Boolean algebra $(K, \times, ')$ has exactly n prime elements, $p, q, r, \ldots$, then it will have exactly 2^n elements in all, namely:

$$Z'; p; q, pq; r, pr, qr, pqr; s, ps, qs, pqs, rs, prs, qrs, pqrs: \quad . \quad . \quad .$$

Proof. By Postulate 1 and Lemma 3, all these 2^n elements exist and are distinct; and by Lemma 7, each element a will find its place somewhere in this list. In particular, $Z =$ the product of all the n primes. (For, if $Z = pq$, say, with r omitted, then $pqr = Zr = Z$, and pqr would not be distinct from pq.) Hence, by Lemma 4, if a is the kth element from one end of the list, then a' will be the kth element from the other end of the list; so that no further elements are introduced by Postulate 2.

By this Lemma 8, Theorem 59 (2) is established.

§6. Introductory Properties of a^* and $a \dashv b$

The theorems of this section, 61–69, depend only on Postulates 1–14; their chief interest, however, is to serve as an introduction to the theorems of Part II.

61. Definition. $a^*. = .a \equiv Z$. (Read: a star.)

In particular, $Z^*. = .Z \equiv Z$.

62. Definition. $a \dashv b. = .(ab')^*$. (Read: a hook b.)

62a. $a \dashv b. = .ab' \equiv Z$. (By 61, 62.)

62b. $a \dashv b$ in $T. \rightleftarrows .a \leqslant b$. (By 43, 62a.)

63 (1). $a \dashv a'. = .a^*$. (By 62, 36.)
(2). $a \dashv Z. = .a^*$. (By 62, 35.)

64 (1). Z^* is in T. (By 61, A, 26.)
(2). a^* in $T. \rightarrow .a = Z$. (By 61, A.)
(3). $a = b. \rightarrow .a^* = b^*$. (By 61, 10.)
(4). $a^* = Z^*. \rightarrow .a = Z$. (By 64 (1), 64 (2).)

65 (1). $a \dashv a. = .Z^*$. (By 62, 34.)
(2). $Z \dashv a. = .Z^*$. (By 62, 37.)
(3). $a \dashv Z'. = .Z^*$. (By 62, 31, 37.)

66. $ab \dashv a. = .Z^*$. (By 62, 13, 34, 37.)

67. $a \dashv ab. = .a \dashv b.$

Proof. By 62, 31, 39, 34, 35, 31, 62,
$a \dashv ab \ . = . \ [a(a''b'')']^* \ . = . \ [(aa')'(ab')']'^* \ . = . \ [(ab')']'^* \ . = . \ (ab')^* \ . = . \ a \dashv b.$

68. $a \dashv b. = .b' \dashv a'.$

Proof. By 62, 31, 12, 62, $a \dashv b \ . = . \ (ab')^* \ . = . \ (b'a'')^* \ . = . \ b' \dashv a'.$

69. $(a \dashv b)$ in $T \ .\longrightarrow. \ (a \dashv b) = Z^*.$

Proof. By 62a, 64 (3), 62,
$(a \dashv b)$ in $T \ .\longrightarrow. \ ab' = Z \ .\longrightarrow. \ (ab')^* = Z^* \ .\longrightarrow. \ (a \dashv b) = Z^*.$

This theorem 69, in view of 64 (1), shows that among all the elements of T, the only one which can appear in a table for $a \dashv b$ is the element Z^*. (Compare 71.)

§7. Theorems Requiring the Use of Postulate 15.

The preceding theorems have all been proved on the basis of Part I, that is, Postulates 1–14. We now turn to the theorems of Part II, which require for their proof one or more of the Postulates 15–17, which are characteristic of Lewis's theory of strict implication.

Theorems 71–75 depend on 1–14 and 15. They cannot be proved from 1–17 if 15 is omitted. (See Example 15.)

71. $(a \equiv b)$ in $T. \longrightarrow .(a \equiv b) = Z^*.$

Proof. By A, 8, 34, 64 (3), 62,
$(a \equiv b)$ in $T. \ \longrightarrow .a = b. \longrightarrow .ab' = bb'. \longrightarrow .(ab')^* = Z^*. \longrightarrow .(a \dashv b) = Z^*.$
Hence by 11, $(a \equiv b)$ in $T \ .\longrightarrow .(b \dashv a) = Z^*.$ Hence by 28a, 15, 36,
$(a \equiv b)$ in $T \ .\longrightarrow .(a \dashv b)(b \dashv a) = Z^*Z^*. \longrightarrow .(a \equiv b) = Z^*.$

This theorem 71, in view of 64 (1), shows that among all the elements of the subclass T, the only one which can appear in a table for $a \equiv b$ is the element Z^*. (Compare 69.)

72. $a \equiv a. = .Z^*.$

Proof. By 15, 65 (1), 36, $a \equiv a. = .(a \dashv a)(a \dashv a). = .Z^*Z^*. = .Z^*.$

73. $a^*Z^*. = .a^*.$

Proof. By 63 (2), 65 (2), 15, 61, $a^*Z^*. = .(a \dashv Z)(Z \dashv a). = .a \equiv Z. = .a^*.$

74. $a \dashv b. = .ab \equiv a.$

Proof. By 15, 66, 67, 62, 73, 62,
$(ab \equiv a) \ . = . \ (a \dashv ab)(ab \dashv a) \ . = . \ (a \dashv b) \ Z^*. = . \ (ab')^*Z^*. = .(ab')^*. = .a \dashv b.$

This important theorem establishes the complete equivalence between Lewis's "strict" implication $(ab' \equiv Z)$ and our "Boolean" implication $(ab \equiv a)$. While Theorem 44: $(a \lesseqgtr b. \rightleftarrows .ab = a)$ is an immediate consequence of Theorem 74: $(a \prec b. = .ab \equiv a)$, it should be noted that 74 cannot be deduced from 44 without the aid of 15.

75. $a^* \prec Z^*. = .Z^*.$

Proof. By 74, 73, 72, $a^* \prec Z^*. = .a^*Z^* \equiv a^*. = .a^* \equiv a^*. = .Z^*.$

The following theorem 78 depends on 1–14 and 15 and 16. It cannot be proved from 1–17 without 15; and it cannot be proved from 1–17 without 16. (See Examples 15, 16.)

78. $a^* \prec (ab)^*. = .Z^*.$

Proof. By 65 (2), $Z \prec a = Z \prec b'$. Hence by 61, 15, $a^*. = .a \equiv Z. = .(a \prec Z)(Z \prec a). = .(a \prec Z)(Z \prec b')$. By 62, 31, $(ab)^*. = .(a \prec b')$. Hence, $a^* \prec (ab)^*. = .(a \prec Z)(Z \prec b') \prec (a \prec b')$, which by 16 is in T. Hence, $a^* \prec (ab)^*$ is in T. Hence by 69, $a^* \prec (ab)^*. = .Z^*.$

§8. Theorems Requiring the Use of Postulate 17

The six following theorems, 81–86, depend on Postulates 1–14 and 17. They cannot be proved from 1–17 without the use of 17. (See Example 17.)

81. $a(a \prec b) \prec b. = .Z^*.$ (By 17, 69.)
82. $a^*a = Z.$

Proof. By 36, 12, 13, 31, 62, 63 (1), 81,
$(a^*a)^*. = .[a^*(aa)]^*. = .[(aa^*)a'']^*. = .aa^* \prec a'. = .a(a \prec a') \prec a'. = .Z^*.$
Hence by 64 (4), $a^*a = Z$.

83. $a^* \prec a'. = .Z^*.$ (By 62, 31, 82.)
84. $Z'^* = Z.$

Proof. By 17, $Z'(Z' \prec Z) \prec Z$ is in T. Hence by 35, $(Z' \prec Z) \prec Z$ is in T. Hence by 63 (2), $Z'^* \prec Z$ is in T. Hence by 63 (2), Z'^{**} is in T. Hence by 64 (2), $Z'^* = Z$.

85. $Z' \prec Z. = .Z.$ (By 62, 36, 84.)
86. $a(a^{*\prime}) = a.$

Proof. By 43, 44, $ab' = Z. \rightarrow .ab = a$. Hence, $a(a^{*\prime})' = Z. \rightarrow .a(a^{*\prime}) = a$. But by 31, 82, $a(a^{*\prime})' = aa^* = Z$. Hence, $a(a^{*\prime}) = a$.

The following Theorem 87 depends on 1–14 and 15 and 17. It cannot be proved from 1–17 without 15; and it cannot be proved from 1–17 without 17. (See Examples 15, 17.)

87. $Z^{**} = Z$.

Proof. By 61, 15, 63 (2), 65 (2), 82,
$Z^{**} = .Z^* \equiv Z. = .(Z^* \hookleftarrow Z)(Z \hookleftarrow Z^*). = .Z^{**}Z^*. = .Z.$

§9. Lewis's Postulates on the Base (K, T, $\times$, $'$, *).

In Lewis's own presentation of his theory, the "undefined ideas" are, in our notation, K, T, $\times$, $'$, *. Here K is a class of elements $a, b, c, \ldots$; T is a subclass in K; $\times$ (read "times") is a binary operator; $'$ (read "dash") is a unary operator; and * (read "star") is another unary operator. [The binary operator which we have denoted by $\equiv$ (read "quad") is introduced by definition, as is the binary operator $\hookleftarrow$ (read "hook").]

The relation of interchangeability, $a = b$, is treated as part of ordinary language, like $\rightarrow$, $\&$, and $\exists$. The properties of the equality sign " = " which we have deduced in Theorems 26, 27, 28 are treated as part of the language, and our Theorem 29 (the "Theorem of Replacement") is postulated as a rule of procedure [Lewis and Langford, Symbolic Logic, page 125, (b)]. The Principle of Substitution for variables is of course assumed [page 125, (a)], as well as (tacitly) the usual postulates of closure [page 189, footnote].

Besides the principle of substitution, Lewis's list, in our notation, is as follows (on the base K, T, $\times$, $'$, *):

Postulate 1. a in $K.\&.b$ in $K :\rightarrow : ab$ in K. [By 1.]
Postulate 2. a in $K.\rightarrow.a'$ in K. [By 2.]
Postulate 3a. a in $K.\rightarrow.a^*$ in K. [By 3, Def. C.]
Postulate 4 (a). a in $T.\rightarrow.a$ in K. [By 4.]
(b). $\exists\, a:(a$ in $T)$.
(c). $\exists\, a:(a$ in $K.\&.a$ not in $T)$.
Definition Df2. $a \hookleftarrow b. = .(ab')^*$. [By Def. D.]
Definition Df3. $a \equiv b. = .(a \hookleftarrow b)(b \hookleftarrow a)$. [By 15.]

Rule of Replacement. $(a \hookleftarrow b)(b \hookleftarrow a)$ in $T :\rightleftarrows : a = b$.

That is, if $(a \hookleftarrow b)(b \hookleftarrow a)$, or $a \equiv b$, is in T, then either of the two elements a and b may be replaced by the other at any point of any formula of the system. [By Def. A and Th. 29.]

Rule of Adjunction. a in $T.\&.b$ in $T :\rightarrow : ab$ in T. [By 5.]
Rule of Inference. a in $T.\&.a \hookleftarrow b$ in $T :\rightarrow : b$ in T. [By 62(b), 49.]

Postulate B1. $ab \hookleftarrow ba$ in T. [By 12, 65 (1), 64 (1).]
Postulate B2. $ab \hookleftarrow a$ in T. [By 66, 64 (1).]
Postulate B3. $a \hookleftarrow aa$ in T. [By 36, 65 (1), 64 (1).]

POSTULATE B4. $(ab)c \prec a(bc)$ in T. [By 13, 65 (1), 64 (1).]
POSTULATE B5. $a \prec a''$ in T. [By 31, 65 (1), 64 (1).]

POSTULATE B6. $[(a \prec b)(b \prec c) \prec (a \prec c)]$ in T. [By 16.]
POSTULATE B7. $[a(a \prec b) \prec b]$ in T. [By 17.]
POSTULATE B8. $a^* \prec (ab)^*$ in T. [By 78, 64 (1).]

POSTULATE B9. $\exists (a, b):[(a \prec b)'(a \prec b')']'$ not in T. [See Note 1.]

Note 1. The existence postulate B9 in Lewis's list is not included among our postulates 1–17, since it is not required for the proofs of any of our theorems, and since there is a certain advantage in leaving "existence" questions open. It must be added to our list, however, if our system is to agree strictly with Lewis's. The form in which B9 is here given is the form now preferred by Professor Lewis; cf. §10, below.

Note 2. Postulate B5 is redundant, since J. C. C. McKinsey has shown that it can be deduced from Df2, Df3, B1, B2, B3, B6 (Bull. Amer. Math. Soc., vol. 40, pp. 425–427, 1934). See 93.1, below.

Note 3. As to the form here given to Lewis's "consistency postulate" B8, a closer rendition of Lewis's B8 would be $(ab)^{*\prime} \prec a^{*\prime}$ in T. The two forms are equivalent by Lewis's theorem 12.44 (our 93.2, below).

Note 4. Postulate B8, though a theorem in our system (with base K, T, $\times$, $'$, $\equiv$), is an independent postulate in Lewis's list (with base K, T, $\times$, $'$, *). Postulate B8 may be replaced, however, by the following equivalent form:

POSTULATE B8a. $a^*.=.a \equiv Z$,

as is proved in 99, below.

All Lewis's postulates (except B9) are deducible from our list 1–17, as indicated in brackets after each item, above.

All our postulates 1–17 (including Definitions A–D) are deducible from Lewis's list (without using B9). This is established by the following theorems, 91–98,—several of which are not explicitly mentioned in Lewis's book.

For brevity, we first introduce the following definition (read: a implies b):

90.0. DEFINITION. $a \leqslant b.\rightleftarrows.a \prec b$ in T; (Our 62b.)
and then establish three lemmas, as follows:

90.1. LEMMA 1. $a \leqslant b.\&.b \leqslant a:\rightarrow:a = b.$

Proof by the rules of Adjunction (Adj.) and Replacement (Repl.).

90.2. LEMMA 2. $a \leqslant ab :\rightarrow: a = ab$.

Proof. By Adj. and Repl., $a \leqslant ab.\,\&.\,ab \leqslant a :\rightarrow: a = ab$. But by B2, $ab \leqslant a$ always holds. Hence, $a \leqslant ab :\rightarrow: a = ab$.

90.3. LEMMA 3. $a \leqslant b.\,\&.\,b \leqslant c :\rightarrow: a \leqslant c$.

Proof. By Adj., $a \leqslant b.\,\&.\,b \leqslant c :\rightarrow: (a \dashv b)(b \dashv c)$ in T. By B6. $(a \dashv b)(b \dashv c) \dashv (a \dashv c)$ in T, whence by the Rule of Inference, $(a \dashv b)(b \dashv c)$ in $T :\rightarrow: (a \dashv c)$ in T. Hence $a \leqslant b.\,\&.\,b \leqslant c :\rightarrow: a \dashv c$ in T.

(A fourth and fifth lemma will be proved in 94.4, 94.5, below.)

In the following deductions from Lewis's list, the theorems numbered from 91 to 96 depend only on Postulates B1–B6.

91.1. $a \leqslant a$; that is, $a \dashv a$ in T.

Proof. By B3, B2, $a \leqslant aa$ and $aa \leqslant a$. Hence by 90.3, $a \leqslant a$.

91.2. $a = aa$.

Proof. By B2, B3, $aa \leqslant a$ and $a \leqslant aa$. Hence by 90.1, $a = aa$.

92.1. $ab = ba$.

Proof. By B1, $ab \leqslant ba$. By B1, $ba \leqslant ab$. Hence by 90.1, $ab = ba$.

92.2. $(ab)c = a(bc)$.

Proof. By B4, $(cb)a. \leqslant .c(ba)$. By 92.1, $(cb)a. = .(bc)a. = .a(bc)$.
Hence, $a(bc). \leqslant .c(ba)$. By 92.1, $c(ba). = .(ba)c. = .(ab)c$.
Hence, $a(bc). \leqslant .(ab)c$. But by B4, $(ab)c. \leqslant .a(bc)$.
Hence by 90.1, $(ab)c = a(bc)$.

93.1. $a'' = a$.

Proof (after McKinsey). By Df2, 92.1, $a' \dashv a'. = .(a'a'')^*. = .(a''a')^* . = .a'' \dashv a$. By 91.1, $a' \dashv a'$ in T. Hence, $a'' \dashv a$ in T. (1)
By (1), $a''' \dashv a''$ in T. Hence by 90.3, 90.0, $a''' \dashv a$ in T.
By Df2, 92.1, $a''' \dashv a. = .(a''' a')^*. = .(a'a''')^*. = .a' \dashv a'''$.
Hence, $a' \dashv a'''$ in T. By (1), $a''' \dashv a'$ in T. Hence by 90.1, $a''' = a'$.
Hence, by Df2, $a \dashv a''. = .(aa''')^*. = .(aa')^*. = .a \dashv a$. By 91.1, $a \dashv a$ in T
Hence, $a \dashv a''$ in T. (2)
From (1) and (2), by 90.1, $a'' = a$. (From this theorem, Postulate B5 follows as a corollary, by 91.1.)

93.2. $\sim a \dashv b. = .b' \dashv a'$.

Proof. By Df2, 92.1, 93.1, $a \dashv b. = .(ab')^*. = .(b'a'')^*. = .b' \dashv a'$.

94.1. $aa' = bb'$, whence DEFINITION B: $Z = aa'$. (Not in the Lewis and Langford book.)

Proof. By Df2, 92, $aa' \dashv c. = .[(aa')c']^* . = .[(ac')a']^* . = .ac' \dashv a$. By B2, $ac' \dashv a$ in T. Hence, $aa' \dashv c$ in T. Hence, $aa' \dashv bb'$ in T and $bb' \dashv aa'$ in T. Hence by 90.1, $aa' = bb'$.

94.2. $a \dashv a = Z^*$, and Z^* in T.

Proof. By B2, 94.1, $a \dashv a. = .(aa')^* . = .Z^*$. By 91.1, $a \dashv a$ in T. Hence, Z^* in T.

94.3. $aZ = Z$.

Proof. By 94.1, 92.2, 91.2, $aZ. = .a(aa'). = .(aa)a'. = .aa'. = .Z$.

94.4. LEMMA 4. $a \leqslant b. \rightarrow .ac \leqslant bc$.

Proof. By B6, $(cd' \dashv b')(b' \dashv a') \dashv (cd' \dashv a')$ in T. (1)
By Df2, 93.1, 92, $(cd' \dashv b'). = .[(cd')b]^* . = .[(bc)d']^* . = .(bc \dashv d)$ (a)
By 93.2, $b' \dashv a' = b \dashv a$. (b)
By Df2, 93.1, 92, $(cd' \dashv a'). = .[(cd')a]^* . = .[(ac)d']^* . = .(ac \dashv d)$. (c)
From (a), (b), (c) in (1), $(bc \dashv d)(a \dashv b) \dashv (ac \dashv d)$ in T. (2)
Hence, putting bc for d and using 94.2, $Z^*(a \dashv b) \dashv (ac \dashv bc)$ in T.
Hence, by Adjunction and Inference,
Z^* in $T . \&. a \dashv b$ in $T : \rightarrow : ac \dashv bc$ in T.
But by 94.2, Z^* in T holds. Hence, $a \dashv b$ in $T : \rightarrow : ac \dashv bc$ in T.

94.5. LEMMA 5. $a \leqslant b. \&. a \leqslant c : \rightarrow : a \leqslant bc$.

Proof. By 94.4, 92.1, 90.3, 91.2, $a \leqslant b. \&. a \leqslant c : \rightarrow : aa \leqslant ab. \&. ab \leqslant bc : \rightarrow : aa \leqslant bc : \rightarrow : a \leqslant bc$.

94.6. $aZ' = a$. (Not in the book.)

Proof. By Df2, 93.1, 94.3, $a \dashv Z'. = .(aZ'')^* . = .(aZ)^* . = .Z^*$. By 94.2, Z^* in T. Hence, $a \dashv Z'$ in T; that is, $a \leqslant Z'$. Hence by 94.4, $aa \leqslant aZ'$, whence by 91.2, $a \leqslant aZ'$. Hence by 90.2, $aZ' = a$.

95.1. $a(ab)' = ab'$.

Proof. By Df2, 93.1, 92, 94.1, $a(ab)' \dashv b'. = .[a(ab)'b]^* . = .[(ab)(ab)']^* . = .Z^*$, whence by 94.2, 90.0, $a(ab)' \leqslant b'$. Hence by 94.4, 91.2, $a(ab)'. \leqslant .ab'$. (1)
By B2, 93.2, $b' \leqslant (ab)'$. By B2, 92.1, $ab' \leqslant b'$. Hence by 90.3, $ab' \leqslant (ab)'$. Hence by 94.4, 91.2, $ab'. \leqslant .a(ab)'$. (2)
From (1), (2), by 90.1, $a(ab)' = ab'$.

95.2. $a \dashv b. = .a \dashv ab$.

Proof. By Df2, 95.1, $a \dashv b. = .(ab')^* . = .[a(ab)']^* . = .a \dashv ab$.

95.3. $a(a'b')' = a$.

Proof. By B2, $a'b' \leqslant a'$. Hence by 93.2, 93.1, $a \leqslant (a'b')$.
Hence by 94.4, 91.2, $a \leqslant a(a'b')'$. Hence by 90.2, $a(a'b')' = a$.

96.1. $(a'b)'(a'b')' = a$. [Our 14.]

Proof. By B2, 93.2, $a \leqslant (a'b)'$ and $a \leqslant (a'b')'$.
Hence by 94.5, 91.2, $a \leqslant (a'b)'(a'b')'$. (1)
By Df2, 95.1, 94.1, $(a'b)'(a'b')' \dashv a. = .[(a'b)'(a'b')'a']^*$.
$= .[(a'b)'(a'b)]^* = .Z^*$, whence by 94.2, $(a'b)'(a'b')' \leqslant a$. (2)
From (1), (2), by 90.1, $(a'b)'(a'b')' = a$.

96.2. $a(b'c')' = [(ab)'(ac)']'$. (Proof as for our 39.)

The next two theorems, 97 and 98, require the use of B8. That they cannot be proved from Lewis's list without the use of B8 is shown by Example B8, below.

97. $a \dashv b. = .ab \equiv a$. (Our 74.) (Not in the book.)

Proof. (As indicated in the footnote at the beginning of this paper, the proof of 97 from Lewis's postulates follows as a corollary from my paper of 1934. For a valuable suggestion in connection with the following more direct proof, I am indebted to W. T. Parry.)
By B8, $a^* \dashv (aa')^*$ in T. Hence by 94.1, $a^* \leqslant Z^*$. (1)
By Df2, $a \dashv ab. = .[a(ab)']^*$, and by Df2, 92, 94.3, $ab \dashv a. = .[aba']^*$.
$= .Z^*$. Hence by (1), $a \dashv ab. \leqslant .ab \dashv a$. Hence by 94.4, 91.2,
$a \dashv ab. \leqslant .(a \dashv ab)(ab \dashv a)$, whence by 90.2, $a \dashv ab. = .(a \dashv ab)(ab \dashv a)$.
Hence by 95.2, Df3, $a \dashv b. = .ab \equiv a$.

98. $a^* = .a \equiv Z$. (Not in the book.) (Our Def. C.)

Proof. By 91.2, 93.1, Df2, 97, 94.1,
$a^* = .(aa'')^* = .a \dashv a' = .aa' \equiv a. = .Z \equiv a. = .a \equiv Z$.

These theorems 91–98 show that all our Postulates 1–17, including Definitions A–D, are deducible as theorems from Lewis's list, without using B9. *Thus the complete equivalence of our Postulates 1–17 and Lewis's Postulates B1–B8 is established.*

The following Theorem 99 is added to show that Theorem 98 may be taken as a postulate in Lewis's system, in place of Postulate B8. Thus, if we adopt

POSTULATE B8a. $a^* = .a \equiv Z$, where $Z = aa'$,

then B8 becomes a theorem, as follows.

99. $a^* \dashv (ab)^*$ in T.

Proof (by B1–B6 and B8a). By Df2, 94.3, $Z \dashv a. = .(Za')^* = .Z^*$.
Hence $Z \dashv b'. = .Z \dashv a$. By B8a, Df3, $a^* = .a \equiv Z. = .(a \dashv Z)(Z \dashv a)$.

Hence $a^* .=. (a \dashv Z)(Z \dashv b')$. By Df2, 93.1, $(ab)^* .=. (a \dashv b')$.
Hence $a^* \dashv (ab)^* .=. (a \dashv Z)(Z \dashv b') \dashv (a \dashv b')$.
Hence by B6, $a^* \dashv (ab)^*$ in T.

Theorems 98 and 99 together show that Postulates B8 and B8a are interchangeable in Lewis's list.[9]

§10. Note on Existence Postulates

Lewis's existence postulate B9 is, in our notation:

B9. $\exists\,(a, b)$: $\{[(a \dashv b)'(a \dashv b')']'$ not in $T\}$.

Another existence postulate which suggests itself as a possible addition to the list is

B10. $\exists\, a$: (a not in T .&. a' not in T).

[Here B9 is independent of B1–B8, B10; and B10 is independent of B1–B9; as is shown by Examples B9 and B10, below.] The suggested postulate B10, while not technically included in Lewis's list, would appear to be entirely in keeping with the spirit of his theory.

Note 1. The contradictory of B9, which we may call B9′, is

B9′. $[(a \dashv b)'(a \dashv b')']'$ in T.

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[9] On page 151 of "Symbolic Logic," Lewis raises the question whether his theorems 19.6, 19.61, 19.62 can be proved from B1–B7, without the use of B8. Our examples B8a and B8 show that 19.6 and 19.62 cannot be so proved. The question in regard to 19.61 is still open. (It has been shown, however, by one of my students, Mr. H. M. Syer, that *if* an "example" for 19.61 exists, it must contain at least sixteen elements.) For convenience of reference, we append direct proofs of these theorems, on the basis of B1–B8.

19.6. $(a \dashv b) \dashv (ac \dashv bc)$ in T.

Proof. By Df2, $a \dashv b = (ab')^*$. By B8, $(ab')^* \dashv (acb')^*$.
By 94.3, 94.6, 93.1, 96.2,
$(acb')^* = [(acb')'(acc')']'^* = [(ac)(bc)']^* = ac \dashv bc$. Hence the theorem.

19.61. $(a \dashv b)(a \dashv c) \dashv (a \dashv bc)$ in T.

Proof. By 19.6, $(a \dashv b) \dashv (ac \dashv bc)$ in T. Hence by 19.6, 95.2,
$(a \dashv b)[a \dashv c] \dashv (ac \dashv bc)[a \dashv ac]$ in T. By 92.1, B6,
$(ac \dashv bc)[a \dashv ac] \dashv (a \dashv bc)$ in T. Hence the theorem, by B6 and Inf.

19.62. $(a \dashv bc) \dashv (a \dashv b)(a \dashv c)$ in T.

Proof. By B8, $\{[(ab')'(ac')']'\}^* \dashv \{(ab')[(ab')'(ac')']'\}^*$ in T.
Hence by 96.2, 95.3, $[a(bc)']^* \dashv (ab')^*$ in T. Hence by Df2,
$(a \dashv bc) \dashv (a \dashv b)$in T; and by 92.1, $(a \dashv bc) \dashv (a \dashv c)$ in T.
Hence the theorem, by Adj.

The contradictory of B10, which we may call B10′, is

B10′. a not in $T .\longrightarrow. a'$ in T.

If we regard " $\prec$ " for the moment as an undefined symbol, we observe that B9′ is a characteristic formula of the Whitehead-Russell theory of "material implication"; and that B10′, while not technically a part of the theory of material implication, is entirely in keeping with the spirit of that theory. (In fact, B10′, in the form "Either a is in T or a' is in T," is, in effect, a part of the "informal" presentation of the theory in Principia Mathematica.)

Note 2. A system $(K, T, \times, ', *)$ which satisfies B1–B10 may or may not have its Z^* distinct from its Z'. If such a system does have its Z^* distinct from its Z', then it must contain at least eight elements.

Note 3. Lewis's postulate B9 must not be confused with the following weaker postulate B11:

B11. $\exists\,(a, b)$: $[(a \prec b)$ not in $T\ .\&.\ (a \prec b')$ not in $T]$.

This relation B11 is a consequence of B1–B8, and is therefore redundant.

Note 4. Lewis's postulate B9 must not be confused with the following stronger postulate F:

F. $\exists\,(a, b)$: $[(a \prec b)'$ in $T\ .\&.\ (a \prec b')'$ in $T]$.

[Here F is independent of B1–B10. Also, B10 is independent of B1–B9 and F. But B9 is deducible from B1–B8 and F, so that F (in conjunction with B1–B8) is "stronger" than B9.] *This postulate F is not a part of Lewis's theory.* Also, the contradictory of F, namely:

F′. $(a \prec b)'$ in $T .\longrightarrow. (a \prec b')'$ not in T,

is not a characteristic formula of material implication, since it may coexist with B1–B10 (see Example F, below).

§11. Examples for Consistency of Postulates 1–17

The following examples of systems $(K, T, \times, ', \equiv)$ satisfy all the Postulates 1–17. Any one of these examples establishes the consistency of the postulates. The examples with eight and sixteen elements are included for reference, because they are rather troublesome to reproduce and exhibit very clearly the flexibility of the Lewis theory.

In the double-entry tables it is understood that a is in the left-hand margin, and b along the top.

EXAMPLE 0.1. K = a class of four elements, denoted by 1, 2, 3, 4. T = the subclass of two elements, 1, 2. ab, a', and $a \equiv b$ as in the accompanying tables. (Tables for a^* and $a \prec b$ are added for reference; see Defs. C, D.).

ab	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

a	a'
1	4
2	3
3	2
4	1

$a \equiv b$	1	2	3	4
1	2	4	4	4
2	4	2	4	4
3	4	4	2	4
4	4	4	4	2

a	a^*
1	4
2	4
3	4
4	2

$a \prec b$	1	2	3	4
1	2	4	4	4
2	2	2	4	4
3	2	4	2	4
4	2	2	2	2

Here $Z = 4$, $Z' = 1$, and $Z^* = 2$.

EXAMPLE 0.2. K, T, ab, a' as in Ex. 0.1; $a \equiv b$ as in table.

$a \equiv b$	1	2	3	4
1	1	4	3	4
2	4	1	4	3
3	3	4	1	4
4	4	3	4	1

a	a^*
1	4
2	3
3	4
4	1

$a \prec b$	1	2	3	4
1	1	4	3	4
2	1	1	3	3
3	1	4	1	4
4	1	1	1	1

Here $Z = 4$, $Z' = 1$, but $Z^* = 1$.

EXAMPLE 0.3. K, T, ab, a' as in Ex. 0.1; $a \equiv b$ as in table.

$a \equiv b$	1	2	3	4
1	1	4	4	4
2	4	1	4	4
3	4	4	1	4
4	4	4	4	1

a	a^*
1	4
2	4
3	4
4	1

$a \prec b$	1	2	3	4
1	1	4	4	4
2	1	1	4	4
3	1	4	1	4
4	1	1	1	1

Here $Z = 4$, $Z' = 1$, $Z^* = 1$.

These three examples, taken from Lewis and Langford's Symbolic Logic, page 493, can easily be shown to be the only four-element systems which satisfy all the postulates 1–17 (except that in Eaxmples 0.2 and 0.3, "$T = 1, 2$" may be replaced by "$T = 1$".)

In all three of these examples, we have the following "relation-tables," in which a dot means "yes" and the absence of a dot means "no."

a in T.		
1	.	
2	.	
3		
4		

ab in T.	1	2	3	4
1	.	.		
2	.	.		
3				
4				

a' in T.		
1		
2		
3	.	
4	.	

$a \equiv b$ in T.	1	2	3	4
1	.			
2		.		
3			.	
4				.

a^* in T.		
1	.	
2		
3		
4	.	

$a \prec b$ in T.	1	2	3	4
1	.			
2	.	.		
3	.		.	
4	.			.

EXAMPLE 0.4. K = 1, 2, 3, 4, 5, 6, 7, 8, T = 1, 2, (W. T. Parry, *Mind*, vol. 43, 1934.)

ab	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	2	4	4	6	6	8	8
3	3	4	3	4	7	8	7	8
4	4	4	4	4	8	8	8	8
5	5	6	7	8	5	6	7	8
6	6	6	8	8	6	6	8	8
7	7	8	7	8	7	8	7	8
8	8	8	8	8	8	8	8	8

a	a'
1	8
2	7
3	6
4	5
5	4
6	3
7	2
8	1

$a \equiv b$	1	2	3	4	5	6	7	8
1	2	6	8	8	8	8	8	8
2	6	2	8	8	8	8	8	8
3	8	8	2	6	8	8	8	8
4	8	8	6	2	8	8	8	8
5	8	8	8	8	2	6	8	8
6	8	8	8	8	6	2	8	8
7	8	8	8	8	8	8	2	6
8	8	8	8	8	8	8	6	2

a	a^*
1	8
2	8
3	8
4	8
5	8
6	8
7	6
8	2

$a \dashv b$	1	2	3	4	5	6	7	8
1	2	6	8	8	8	8	8	8
2	2	2	8	8	8	8	8	8
3	2	6	2	6	8	8	8	8
4	2	2	2	2	8	8	8	8
5	2	6	8	8	2	6	8	8
6	2	2	8	8	2	2	8	8
7	2	6	2	6	2	6	2	6
8	2	2	2	2	2	2	2	2

Here $Z = 8$, $Z' = 1$, $Z^* = 2$. Here A gives $a = b$ when and only when a and b are the same. All the postulates 1–17 are found to be satisfied. (All except 13, 14, 16 require merely an inspection of the tables; 16 is verified by an examination of cases. The easiest way to verify 13 and 14 is by replacing the elements 1, 2, 3, 4, 5, 6, 7, 8 by the factors of Sheffer's "Boolean number" 30 in the following order: 1, 2; 3, 6; 5, 10, 15, 30, and interpreting ab as the least common multiple of a and b, and a' as the quotient $30/a$. "Relation tables" are as follows:

a in T.

1	.
2	.
3	
4	
5	
6	
7	
8	

a' in T.

1	
2	
3	
4	
5	
6	
7	.
8	.

$a \equiv b$ in T.

	1	2	3	4	5	6	7	8
1	.							
2		.						
3			.					
4				.				
5					.			
6						.		
7							.	
8								.

a^* in T.

1	
2	
3	
4	
5	.
6	.
7	.
8	.

$a \dashv b$ in T.

	1	2	3	4	5	6	7	8
1	.							
2	.	.						
3	.	.	.					
4	.	.	.	.				
5	.				.			
6	.	.				.		
7	.	.	.				.	
8	.	.	.	.				.

EXAMPLE 0.5. K = 1, 2, 3, 4, 5, 6, 7, 8. $T = 1, 2$.

ab and a' as in Ex. 0.4; and $a \equiv b$ as in the following new table:

$a \equiv b$	1	2	3	4	5	6	7	8
1	2	4	8	8	6	8	8	8
2	4	2	8	8	8	6	8	8
3	8	8	2	4	8	8	6	8
4	8	8	4	2	8	8	8	6
5	6	8	8	8	2	4	8	8
6	8	6	8	8	4	2	8	8
7	8	8	6	8	8	8	2	4
8	8	8	8	6	8	8	4	2

a	a^*
1	8
2	8
3	8
4	6
5	8
6	8
7	4
8	2

$a \dashv b$	1	2	3	4	5	6	7	8
1	2	4	8	8	6	8	8	8
2	2	2	8	8	6	6	8	8
3	2	4	2	4	6	8	6	8
4	2	2	2	2	6	6	6	6
5	2	4	8	8	2	4	8	8
6	2	2	8	8	2	2	8	8
7	2	4	2	4	2	4	2	4
8	2	2	2	2	2	2	2	2

Here $Z = 8$, $Z' = 1$, $Z^* = 2$, as in Ex. 0.4. All the postulates 1–17 are satisfied. In spite of the greater variety in the table for a^*, the "relation tables" for this example are exactly the same as those for Ex. 0.4.

EXAMPLE 0.6. This example contains sixteen elements, and exhibits more clearly than any smaller one can do the characteristic flexibility of the system of strict implication. In particular, the subclass T contains other elements besides Z' and Z^*; and there are many elements which belong neither to T nor to T' (see 53). The basic symbols (K, T, $\times$, $'$, $\equiv$) are specified as follows:

$K = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16.$
$T = 1, 2, 3, 4.$

(In the tables, the places which the zero element 16 should occupy are left blank, in order to exhibit more clearly to the eye the "pattern" of the system.)

ab	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
2	2	2	4	4	6	6	8	8	10	10	12	12	14	14		
3	3	4	3	4	7	8	7	8	11	12	11	12	15		15	
4	4	4	4	4	8	8	8	8	12	12	12	12				
5	5	6	7	8	5	6	7	8	13	14	15		13	14	15	
6	6	6	8	8	6	6	8	8	14	14			14	14		
7	7	8	7	8	7	8	7	8	15		15		15		15	
8	8	8	8	8	8	8	8	8								
9	9	10	11	12	13	14	15		9	10	11	12	13	14	15	
10	10	10	12	12	14	14			10	10	12	12	14	14		
11	11	12	11	12	15		15		11	12	11	12	15		15	
12	12	12	12	12					12	12	12	12				
13	13	14	15		13	14	15		13	14	15		13	14	15	
14	14	14			14	14			14	14			14	14		
15	15		15		15		15		15		15		15		15	
16																

a	a'
1	16
2	15
3	14
4	13
5	12
6	11
7	10
8	9
9	8
10	7
11	6
12	5
13	4
14	3
15	2
16	1

$a \equiv b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	4	8			8	8			12							
2	8	4			8	8				12						
3			4	8			8	8			12					
4			8	4			8	8				12				
5	8	8			4	8							12			
6	8	8			8	4								12		
7			8	8			4	8							12	
8			8	8			8	4								12
9	12								4	8			8	8		
10		12							8	4			8	8		
11			12								4	8			8	8
12				12							8	4			8	8
13					12				8	8			4	8		
14						12			8	8			8	4		
15							12				8	8			4	8
16								12			8	8			8	4

The tables for a^* and $a \dashv b$, by Defs. C and D, are as follows:

a	a^*	$a \dashv b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	16	1	4	8			8	8			12							
2	16	2	4	4			8	8			12	12						
3	16	3	4	8	4	8	8	8	8	8	12		12					
4	16	4	4	4	4	4	8	8	8	8	12	12	12	12				
5	16	5	4	8			4	8			12				12			
6	16	6	4	4			4	4			12	12			12	12		
7	16	7	4	8	4	8	4	8	4	8	12		12		12		12	
8	12	8	4	4	4	4	4	4	4	4	12	12	12	12	12	12	12	12
9	16	9	4	8			8	8			4	8			8	8		
10	16	10	4	4			8	8			4	4			8	8		
11	8	11	4	8	4	8	8	8	8	8	4	8	4	8	8	8	8	8
12	8	12	4	4	4	4	8	8	8	8	4	4	4	4	8	8	8	8
13	16	13	4	8			4	8			4	8			4	8		
14	16	14	4	4			4	4			4	4			4	4		
15	8	15	4	8	4	8	4	8	4	8	4	8	4	8	4	8	4	8
16	4	16	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4

Here $Z = 16$, $Z' = 1$, and $Z^* = 4$. All the postulates 1–17 are found to be satisfied. (Postulate 16 is verified by a somewhat laborious examination of cases. Postulates 13 and 14 are most easily verified by replacing the elements 1, 2, . . . 16 by the factors of the Boolean number 210 in the following order:

1, 2; 3, 6; 5, 10, 15, 30; 7, 14, 21, 42, 35, 70, 105, 210;

and interpreting ab as the least common multiple of a and b, and a' as the quotient $210/a$. The relation "$a \dashv b$ in T" then means: "a contains b as a factor.")

The relation tables for a in T, a' in T, $a \equiv b$ in T, and $a \dashv b$ in T may be readily constructed by the reader.

§12. Examples for Independence

§12a. Examples 1–4

Each of the following examples of systems $(K, T, \times, ', \equiv)$ violates the postulate with corresponding number.

Example 1. $K = 1, 2$. $T = 1$.

ab	1	2
1	1	w
2	w	2

a	a'
1	2
2	1

$a \equiv b$	1	2
1	1	2
2	2	1

Here 1 fails. 2, 3, 4, 5, 6, hold. Def. A gives $1 = 1$, $2 = 2$, so that 7–14 hold in so far as the elements involved are in K.

EXAMPLE 2. $K = 1, 2$. $T = 1$.

ab	1	2
1	1	2
2	2	2

a	a'
1	2
2	w

$a \equiv b$	1	2
1	1	2
2	2	1

Here 2 fails. 1, 3, 4, 5, 6, hold. Def. A gives $1 = 1, 2 = 2$, so that 7–14 hold in so far as the elements involved are in K.

EXAMPLE 3. $K = 1, 2$. $T = 1$.

ab	1	2
1	1	2
2	2	2

a	a'
1	2
2	1

$a \equiv b$	1	2
1	1	w
2	w	1

Here 3 fails. 1, 2, 4, 5, 6 hold. Def. A. gives $1 = 1, 2 = 2$, so that 7–14 hold in so far as the elements involved are in K.

EXAMPLE 4a. $K = 1, 2$. $T = 1, 3$.

ab	1	2
1	1	2
2	2	2

a	a'
1	2
2	1

$a \equiv b$	1	2
1	1	2
2	2	1

Here 4a fails, since the element 3 is in T and not in K. 1, 2, 3, 4b, 4c, hold. 5–14 hold in so far as the elements involved are in K. (The verification of 13 and 14 is accomplished by a simple examination of cases.)

EXAMPLE 4b. $K = 1, 2$. $T =$ an empty class.
ab, a', $a \equiv b$ as in Ex. 4a.

Here 4b fails. 1, 2, 3, hold. 4a holds (vacuously). 4c holds. 5, 6 hold (vacuously). Def. A gives $a = b$ never true, so that 7–10 hold vacuously. 11–14 fail. As noted above, if any one of the postulates 11–14 is included, 4b may be replaced by the weaker form: $\exists\, a$: (a in K).

EXAMPLE 4c. $K = 1, 2$. $T = 1, 2$. ab, a', $a \equiv b$ as in Ex. 4a.

Here 4c fails. 1, 2, 3, 4a, 4b, 5, 6 hold. Def. A gives $a = b$ true for all values, so that 7–14 hold.

These six examples show that Postulates 1, 2, 3, 4a, 4b, 4c are independent of each other. With the exception noted under 4b, they are also independent of 5–14.

§12b. Examples 5–14

Example 5. $K = 1, 2, 3, 4, 5, 6, 7, 8$. $T = 1, 2, 3, 4, 5, 6$.
ab and a' as in Example 0.4 (Boolean algebra); $a \equiv b$ as in table:

$a \equiv b$	1	2	3	4	5	6	7	8
1	1	2	8	8	8	8	8	8
2	2	1	8	8	8	8	8	8
3	8	8	1	2	8	8	8	8
4	8	8	2	1	8	8	8	8
5	8	8	8	8	1	2	8	8
6	8	8	8	8	2	1	8	8
7	8	8	8	8	8	8	1	2
8	8	8	8	8	8	8	2	1

a	a^*
1	8
2	8
3	8
4	8
5	8
6	8
7	2
8	1

$a \prec b$	1	2	3	4	5	6	7	8
1	1	2	8	8	8	8	8	8
2	1	1	8	8	8	8	8	8
3	1	2	1	2	8	8	8	8
4	1	1	1	1	8	8	8	8
5	1	2	8	8	1	2	8	8
6	1	1	8	8	1	1	8	8
7	1	2	1	2	1	2	1	2
8	1	1	1	1	1	1	1	1

Here $Z = 8$, $Z^* = 1$. Postulate 5 fails, since 5 and 4 are in T, while 5×4 [$= 8$] is not in T. Postulates 1–4, 6–14 all hold. (By A, we have not only $1 = 1, 2 = 2, \ldots 8 = 8$, but also $1 = 2, 3 = 4$, $5 = 6$, and $7 = 8$.)

Example 6. $K = 1, 2, 3, 4$. $T = 2$.
ab, a', $a \equiv b$ as in Example 0.1. ($Z = 4$, $Z^* = 2$.)

Here 6 fails, since 1×2 [$= 2$] is in T, while 1 is not in T. Postulates 1–5, 7–14 hold.

Example 7. $K = 1, 2, 3, 4$. $T = 1, 2$.
ab and a' as in Example 0.1. $a \equiv b$ as in table. ($Z = 4$, $Z^* = 1$.)

$a \equiv b$	1	2	3	4
1	1	1	1	1
2	1	1	2	1
3	1	2	1	1
4	1	1	1	1

Here 7 fails, since 2 is in T and $2 \equiv 4$ [$= 1$] is in T, while 4 is not in T. Postulates 1–6, 8–14 hold. (By A, we have $a = b$ for all values of a and b.)

Example 8. $K = 1, 2, 3, 4, 5, 6, 7, 8$. $T = 1, 2$.
ab and a' as in Example 0.4. $a \equiv b$ as in table. ($Z = 8$, $Z^* = 1$.)

$a \equiv b$	1	2	3	4	5	6	7	8
1	1	2	8	8	8	8	8	8
2	2	1	8	8	8	8	8	8
3	8	8	1	8	8	8	8	8
4	8	8	8	1	8	8	8	8
5	8	8	8	8	1	8	8	8
6	8	8	8	8	8	1	8	8
7	8	8	8	8	8	8	1	2
8	8	8	8	8	8	8	2	1

Here 8 fails, since $1 \equiv 2$ [= 2] is in T, while $1 \times 5 \equiv 2 \times 5$ [= 8] is not in T. Postulates 1–7, 9–14 hold. (By A, we have not only $1 = 1, 2 = 2, \ldots 8 = 8$, but also $1 = 2$ and $7 = 8$.)

EXAMPLE 9. $K = 1, 2, 3, 4, 5, 6, 7, 8$. $T = 1, 2$.
ab and a' as in Example 0.4. $a \equiv b$ as in table. ($Z = 8$, $Z^* = 1$.)

$a \equiv b$	1	2	3	4	5	6	7	8
1	1	2	8	8	8	8	8	8
2	2	1	8	8	8	8	8	8
3	8	8	1	2	8	8	8	8
4	8	8	2	1	8	8	8	8
5	8	8	8	8	1	2	2	2
6	8	8	8	8	2	1	2	2
7	8	8	8	8	2	2	1	2
8	8	8	8	8	2	2	2	1

Here 9 fails, since $5 \equiv 7$ is in T, while $5' \equiv 7'$ [= 8] is not in T. By A, we have not only $1 = 1$, $2 = 2$, $\ldots$ $8 = 8$, but also $1 = 2$, $3 = 4$, $5 = 6$, $5 = 7$, $5 = 8$, $6 = 7$, $6 = 8$, $7 = 8$. Postulates 1–8, 10–14, hold.

EXAMPLE 10. $K = 1, 2, 3, 4, 5, 6, 7, 8$. $T = 1, 2$.
ab and a' as in Example 0.4. $a \equiv b$ as in table.

$a \equiv b$	1	2	3	4	5	6	7	8
1	1	2	8	8	8	8	8	8
2	2	1	8	8	8	8	8	8
3	8	8	1	2	8	8	8	8
4	8	8	2	1	8	8	8	8
5	8	8	8	8	1	2	6	8
6	8	8	8	8	2	1	8	8
7	8	8	8	8	6	8	1	2
8	8	8	8	8	8	8	2	1

Here 10 fails, since $5 \equiv 6$ [= 1] is in T, while $(5 \equiv 7) \equiv (6 \equiv 7)$ [= 8] is not in T. By A we have not only $1 = 1, 2 = 2, \ldots 8 = 8$, but also $1 = 2$, $3 = 4$, $5 = 6$, $7 = 8$. Postulates 1–9, 11–14 hold.

EXAMPLE 11. K = 1, 2, 3, 4. $T = 1, 2$.
ab and a' as in Example 0.1. $a \equiv b$ as in table.

$a \equiv b$	1	2	3	4
1	2	4	4	4
2	4	2	4	4
3	4	4	2	4
4	4	4	3	2

Here 11 fails, since $(3 \equiv 4) \equiv (4 \equiv 3)$ [= 3] is not in T. Postulates 1–10, 12–14 hold.

EXAMPLE 12. $K = 1, 2, 3, 4$. $T = 1$.
ab, a', and $a \equiv b$ as in tables.

ab	1	2	3	4
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4

a	a'
1	4
2	3
3	2
4	1

$a \equiv b$	1	2	3	4
1	1	4	4	4
2	4	1	4	4
3	4	4	1	4
4	4	4	4	1

Here 12 fails, since $(3 \times 4) \equiv (4 \times 3)$ [$= 4$] is not in T. Postulates 1–11, 13–14 hold. In this system the theorem $aa' = bb'$ does not hold, and no zero-element Z exists.

EXAMPLE 13. $K = 1, 2, 3, 4$. $T = 1$. ab, a', $a \equiv b$ as in tables:

ab	1	2	3	4
1	1	3	2	4
2	3	3	4	4
3	2	4	2	4
4	4	4	4	4

a	a'
1	4
2	3
3	2
4	1

$a \equiv b$	1	2	3	4
1	1	4	4	4
2	4	1	4	4
3	4	4	1	4
4	4	4	4	1

Here 13 fails, since $(1 \times 2) \times 2 \equiv 1 \times (2 \times 2)$ [$= 4$] is not in T. Postulates 1–12 hold. Postulate 14 holds (by examination of cases).

EXAMPLE 14. $K = 1, 2, 3, 4$. $T = 1, 2$. ab, a', $a \equiv b$ as in tables:

ab	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	4	4
4	4	4	4	4

a	a'
1	4
2	3
3	2
4	1

$a \equiv b$	1	2	3	4
1	2	4	4	4
2	4	2	4	4
3	4	4	2	4
4	4	4	4	2

Here 14 fails, since $(2'3)'\ (2'3')' \equiv 2$ [$= 4$] is not in T. Postulates 1–13 hold (13 by examination of cases).

These Examples 5–14 show that Postulates 5–14 are independent of Postulates 1–4 and of each other.

§12c. EXAMPLES 15–17.

EXAMPLE 15. $K = 1, 2, 3, 4$. $T = 1, 2$.
ab and a' as in Example 0.1. $a \equiv b$ as in table. ($Z = 4$, $Z^* = 2$.)

$a \equiv b$	1	2	3	4
1	1	4	3	4
2	4	1	4	3
3	3	4	2	4
4	4	3	4	2

a	a^*
1	4
2	3
3	4
4	2

$a \dashv b$	1	2	3	4
1	2	4	3	4
2	2	2	3	3
3	2	4	2	4
4	2	2	2	2

$a(a \dashv b)$	1	2	3	4
1	2	4	3	4
2	2	2	4	4
3	4	4	4	4
4	4	4	4	4

$ab \equiv a$	1	2	3	4
1	1	4	3	4
2	1	1	3	3
3	2	4	2	4
4	2	2	2	2

Here 15 fails, since $(1 \dashv 3)\ (3 \dashv 1) \equiv (1 \equiv 3). = .(3)(2) \equiv 3. = .4 \equiv 3. = .4$, which is not in T. Postulates 1–6 hold. Definition A gives $a = b$

when and only when a and b are the same. Postulates 7–14 hold. 16 holds, by examination of cases. 17 holds, by inspection of the table for $a(a \dashv b)$. Hence 15 cannot be deduced from the other postulates of the list 1–17.

We note also that Theorems 71–75, 78, and 87 fail; hence none of these theorems can be deduced from the postulates 1–17 if 15 is omitted.

By way of parenthesis, the following alternative examples, 15a, 15b, may be of interest.

EXAMPLE 15a. $K = 1, 2, 3, 4$. $T = 1, 2$.
ab and a' as in Example 0.1. $a \equiv b$ as in table. ($Z = 4$, $Z^* = 2$.)

$a \equiv b$	1	2	3	4
1	2	4	4	4
2	4	2	3	4
3	4	3	2	4
4	4	4	4	2

a	a^*
1	4
2	4
3	4
4	2

$a \dashv b$	1	2	3	4
1	2	4	4	4
2	2	2	4	4
3	2	4	2	4
4	2	2	2	2

$a(a \dashv b)$	1	2	3	4
1	2	4	4	4
2	2	2	4	4
3	4	4	4	4
4	4	4	4	4

$ab \equiv a$	1	2	3	4
1	2	4	4	4
2	2	2	4	4
3	2	4	2	4
4	2	2	2	2

Here 15 fails, since $(3 \dashv 2)(2 \dashv 3) \equiv (3 \equiv 2). = .(4)(4) \equiv 3. = .4 \equiv 3. = .4$, which is not in T. Postulates 7–14 hold. 16 holds [by examination of cases]. 17 holds [by inspection of table for $a(a \dashv b)$]. Also, all the theorems of §7 and §8 are satisfied. Hence 15 cannot be proved from postulates 1–14, 16–17, even with the aid of §7 and §8.

We note also that a theorem which we may call 15a: $(ab \equiv a)(ba \equiv b) = (a \equiv b)$ fails [when $a = 2$, $b = 3$]. Hence 15a cannot be proved from Postulates 1–14, 16–17, even with the aid of §7 and §8.

EXAMPLE 15b. $K = 1, 2, 3, 4$. $T = 1, 2$.
ab and a' as in Example 0.1. $a \equiv b$ as in table. ($Z = 4$, $Z^* = 1$.)

$a \equiv b$	1	2	3	4
1	1	4	4	4
2	4	1	4	3
3	4	4	1	4
4	4	3	4	1

a	a^*
1	4
2	3
3	4
4	1

$a \dashv b$	1	2	3	4
1	1	4	3	4
2	1	1	3	3
3	1	4	1	4
4	1	1	1	1

$a(a \dashv b)$	1	2	3	4
1	1	4	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

$ab \equiv a$	1	2	3	4
1	1	4	4	4
2	1	1	3	3
3	1	4	1	4
4	1	1	1	1

Here 15 fails, since $(1 \dashv 3)(3 \dashv 1) \equiv (1 \equiv 3). = .4$, which is not in T. Postulates 7–14 hold. 16 holds [by examination of cases]. 17 holds [by inspection of table for $a(a \dashv b)$]. Also, a theorem which we may call 15a: $(ab \equiv a)(ba \equiv b) = (a \equiv b)$ holds [by inspection of table for $ab \equiv a$]. Hence 15 cannot be proved from Postulates 1–14, 16–17, even with the aid of 15a.

We note also that Theorem 74 fails [when $a = 1$ and $b = 3$], while all the other theorems of §7 and §8 hold. Hence 74 cannot be proved from Postulates 1–14, 16–17, even with the aid of all the other theorems of §7 and §8, and 15a.

EXAMPLE 16. $K = 1, 2, 3, 4, 5, 6, 7, 8$. $T = 1, 2$.
ab and a' as in Example 0.4. $a \equiv b$ as in table. ($Z = 8$, $Z^* = 2$.)

$a \equiv b$	1	2	3	4	5	6	7	8
1	2	6	8	4	8	8	8	8
2	6	2	8	8	8	8	8	8
3	8	8	2	6	8	8	8	8
4	4	8	6	2	8	8	8	8
5	8	8	8	8	2	6	8	4
6	8	8	8	8	6	2	8	8
7	8	8	8	8	8	8	2	6
8	8	8	8	8	4	8	6	2

a	a^*
1	8
2	8
3	8
4	8
5	4
6	8
7	6
8	2

$a \dashv b$	1	2	3	4	5	6	7	8
1	2	6	8	4	8	8	8	8
2	2	2	8	8	8	8	8	8
3	2	6	2	6	8	8	8	8
4	2	2	2	2	8	8	8	8
5	2	6	8	4	2	6	8	4
6	2	2	8	8	2	2	8	8
7	2	6	2	6	2	6	2	6
8	2	2	2	2	2	2	2	2

Here 16 fails, since $(3 \dashv 1)(1 \dashv 4) \dashv (3 \dashv 4). = .(2)(4) \dashv 6.. = .4 \dashv 6. = .8$, which is not in T. Postulates 1–14 hold. 15 holds [by inspection of table for $(a \dashv b)(b \dashv a)$]. 17 holds [by inspection of table for $a(a \dashv b)$]. Hence, 16 cannot be deduced from 1–15, 17.

We note also that Theorem 78 fails, since $5^* \dashv (5 \dashv 2). = .4 \dashv 6. = .8$, which is not in T. Hence, 78 cannot be deduced from Postulates 1–17 if 16 is omitted.

The following alternative example, 16a, may also be of interest.

EXAMPLE 16a. $K = 1, 2, 3, 4, 5, 6, 7, 8$. $T = 1, 2$.
ab and a' as in Example 0.4. $a \equiv b$ as in table. ($Z = 8$, $Z^* = 2$.)

$a \equiv b$	1	2	3	4	5	6	7	8
1	2	4	4	8	8	8	8	8
2	4	2	4	4	8	8	8	8
3	4	4	2	4	8	8	8	8
4	8	4	4	2	8	8	8	8
5	8	8	8	8	2	4	4	8
6	8	8	8	8	4	2	4	4
7	8	8	8	8	4	4	2	4
8	8	8	8	8	8	4	4	2

a	a^*
1	8
2	8
3	8
4	8
5	8
6	4
7	4
8	2

$a \dashv b$	1	2	3	4	5	6	7	8
1	2	4	4	8	8	8	8	8
2	2	2	4	4	8	8	8	8
3	2	4	2	4	8	8	8	8
4	2	2	2	2	8	8	8	8
5	2	4	4	8	2	4	4	8
6	2	2	4	4	2	2	4	4
7	2	4	2	4	2	4	2	4
8	2	2	2	2	2	2	2	2

Here 16 fails, since $(5 \dashv 7)(7 \dashv 8) \dashv (5 \dashv 8). = .(4)(4) \dashv 8. = .4 \dashv 8. = .8$, which is not in T. Postulates 1–14 hold. 15 holds [by inspection of tables for $a \dashv b$ and $b \dashv a$]. 17 holds [by inspection of table for $a(a \dashv b)$]. Also, Theorems 71–75, 78, 81–87 hold. Hence, Postulate 16 cannot be deduced from 1–15, 17, even with the aid of §7 and §8.

EXAMPLE 17. $K = 1, 2, 3, 4$. $T = 1, 2$.
ab and a' as in Example 0.1. $a \equiv b$ as in table. ($Z = 4$, $Z^* = 1$.)

$a \equiv b$	1	2	3	4
1	1	3	3	3
2	3	1	3	3
3	3	3	1	3
4	3	3	3	1

a	a^*
1	3
2	3
3	3
4	1

$a \dashv b$	1	2	3	4
1	1	3	3	3
2	1	1	3	3
3	1	3	1	3
4	1	1	1	1

Here 17 fails, since $3(3 \dashv 4) \dashv 4. = .(3)(3) \dashv 4. = .3 \dashv 4. = .3$, which is not in T. Postulates 1–14 hold. 15 holds [by inspection of tables for $a \dashv b$ and $b \dashv a$]. 16 holds [by examination of cases]. Also, Theorems 71–75, 78 hold. Hence, 17 cannot be proved from 1–16, even with the aid of §7.

We note also that 81–87 fail. Hence, 81–87 cannot be deduced from Postulates 1–16 without the aid of 17.

§12d. Examples for B8 in Lewis's List (see §9)

Example B8. Base: $(K, T, \times, ', *)$. $K = 1, 2, 3, 4, 5, 6, 7, 8$. $T = 1, 2$. ab as in table, a' as in table, a^* as in table.

ab	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	2	4	4	6	6	8	8
3	3	4	3	4	7	8	7	8
4	4	4	4	4	8	8	8	8
5	5	6	7	8	5	6	7	8
6	6	6	8	8	6	6	8	8
7	7	8	7	8	7	8	7	8
8	8	8	8	8	8	8	8	8

a	a'
1	8
2	7
3	6
4	5
5	4
6	3
7	2
8	1

a	a^*
1	8
2	8
3	8
4	8
5	8
6	3
7	8
8	2

Here $Z = aa' = 8$; $Z^* = 2$.

In accordance with Definitions Df2 and Df3, we have the following tables for $a \dashv b$ and $a \equiv b$.

$a \dashv b$	1	2	3	4	5	6	7	8
1	2	8	3	8	8	8	8	8
2	2	2	3	3	8	8	8	8
3	2	8	2	8	8	8	8	8
4	2	2	2	2	8	8	8	8
5	2	8	3	8	2	8	3	8
6	2	2	3	3	2	2	3	3
7	2	8	2	8	2	8	2	8
8	2	2	2	2	2	2	2	2

$a \equiv b$	1	2	3	4	5	6	7	8
1	2	8	4	8	8	8	8	8
2	8	2	8	4	8	8	8	8
3	4	8	2	8	8	8	8	8
4	8	4	8	2	8	8	8	8
5	8	8	8	8	2	8	4	8
6	8	8	8	8	8	2	8	4
7	8	8	8	8	4	8	2	8
8	8	8	8	8	8	4	8	2

In this example, B8 and B8a fail. [Take $a = 6$ and $b = 7$.] All the other postulates of Lewis's list are found to be satisfied. [In verifying B4, note that $(ab)c = a(bc)$ as in the example of a Boolean algebra in Example 0.4. B6 is verified by an examination of cases; B7 by inspection of a table for $a(a \dashv b)$. B9 is satisfied by the pair of elements $a = 1$, $b = 7$.] Note, however, that 97 and 98 fail, which shows that 97 and 98 cannot be deduced from B1–B7 without the use of B8 (or B8a). We note further that B10 holds, and that F holds ($a = 3$, $b = 2$). Also, 19.62 fails (see footnote 9, above).

EXAMPLE B8a. K, T, $\times, '$, as in Example B8, with a^* given as follows: $4^* = 7$; $8^* = 2$; otherwise, $a^* = 8$.

In this example, B8 and B8a fail; B1–B7 and B9 hold; and 19.6 fails (see footnote 9, above).

§12e. EXAMPLES B9, B10, AND F (SEE §10).

EXAMPLE B9. $K = 1, 2, 3, 4, 5, 6, 7, 8$. $T = 1$.

ab	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	2	4	4	6	6	8	8
3	3	4	3	4	7	8	7	8
4	4	4	4	4	8	8	8	8
5	5	6	7	8	5	6	7	8
6	6	6	8	8	6	6	8	8
7	7	8	7	8	7	8	7	8
8	8	8	8	8	8	8	8	8

a	a'
1	8
2	7
3	6
4	5
5	4
6	3
7	2
8	1

$a \equiv b$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	5	6
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	5	6	3	4	1	2
8	8	7	6	5	4	3	2	1

a	a^*
1	8
2	7
3	6
4	5
5	4
6	3
7	2
8	1

$a \dashv b$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	1	1	3	3	5	5	7	7
3	1	2	1	2	5	6	5	6
4	1	1	1	1	5	5	5	5
5	1	2	3	4	1	2	3	4
6	1	1	3	3	1	1	3	3
7	1	2	1	2	1	2	1	2
8	1	1	1	1	1	1	1	1

Here B9 fails, since it will be found that $[(a \dashv b)'(a \dashv b')']'$ is in T for all vales of a and b. Postulates 1–17 hold. (15 by inspection, 16 by examination of cases, 17 by inspection); so that B1–B8 also hold. Hence B9 is independent of 1–17, and of B1–B8. (We note that B10 also holds, so that B9 cannot be deduced from B1–B8 even with the aid of B10. On the other hand, F of course fails.)

Since $a^* = a'$, this example is technically an example (in our notation) of a system of material implication. Note, however, that while the formal postulate B9′ is satisfied, the "informal" postulate B10′ is not satisfied in this example.

EXAMPLE B10. K, T, $\times$, $'$, $\equiv$, *, the same as in Example 0.4, except that $T = 1, 2, 3, 4$ instead of $T = 1, 2$.

Here B10 clearly fails. Postulates 1–17 and B1–B9 all hold. Hence B10 is independent of 1–17 and of B1–B9.

(We note that F also holds, taking $a = 2$, $b = 3$; so that B10 cannot be deduced from B1–B9 even with the aid of F.)

EXAMPLE F. $K = 1, 2, 3, 4, 5, 6, 7, 8$. $T = 1, 2$. ab and a' as in Example B9. $a \equiv b$ and a^* and $a \prec b$ as in the following tables:

$a \equiv b$	1	2	3	4	5	6	7	8
1	2	4	4	4	6	8	8	8
2	4	2	4	4	8	6	8	8
3	4	4	2	4	8	8	6	8
4	4	4	4	2	8	8	8	6
5	6	8	8	8	2	4	4	4
6	8	6	8	8	4	2	4	4
7	8	8	6	8	4	4	2	4
8	8	8	8	6	4	4	4	2

a	a^*
1	8
2	8
3	8
4	6
5	4
6	4
7	4
8	2

$a \prec b$	1	2	3	4	5	6	7	8
1	2	4	4	4	6	8	8	8
2	2	2	4	4	6	6	8	8
3	2	4	2	4	6	8	6	8
4	2	2	2	2	6	6	6	6
5	2	4	4	4	2	4	4	4
6	2	2	4	4	2	2	4	4
7	2	4	2	4	2	4	2	4
8	2	2	2	2	2	2	2	2

Here F fails, since $(a \prec b)'(a \prec b')'$ is found to be never in T. Postulates 1–17 hold (16 by examination of cases); so that B1–B8 also hold. B9 holds (taking $a = 3$, $b = 2$). B10 holds. Hence F cannot be deduced from B1–B10.

§13. COMPARISON WITH "MATERIAL IMPLICATION"

We conclude this paper with a strict formulation of the difference between the system of Postulates 1–17 and the system of "material implication."

The theory of "material implication" in Section A of Whitehead and Russell's Principia Mathematica may be regarded as an "abstract deductive theory" dealing with (1) a class of objects p, q, r, . . . ("elementary propositions"), which we may call the class K; (2) the assertion sign, $\vdash$; (3) a unary operator, $\sim$ (read "curl"); (4) a binary operator, $\vee$ (read "wedge"); and three other binary operators, defined in terms of $\sim$ and $\vee$, namely $\supset$ (read "horseshoe"), $\times$ (read "times"), and $\equiv$ (read "triple").

[For typographical convenience we shall replace the prefix "$\vdash$" by the suffix "in T," and also the "curl," $\sim$, by the "dash," $'$; so that "$\vdash .p$" will appear as "p in T," and $\sim p$ will appear as p'.]

Postulates $*1.7$, $*1.71$, $*1.01$, $*3.01$, $*4.01$ of Principia Mathematica[10] show that p', $p \vee q$, $p \supset q$, pq, and $p \equiv q$ are elements of K

[10] The propositions of Principia Mathematica cited in this paper are the following:

$*1.1$. Anything implied by a true elementary proposition is true.

$*1.7$. If p is an elementary proposition, p' is an elementary proposition.

$*1.71$. If p and q are elementary propositions, $p \vee q$ is an elementary proposition.

$*1.01$. $p \supset q . = . p' \vee q$ Df. $*3.01$. $pq . = . (p' \vee q')'$ Df.

$*4.01$. $p \equiv q . = . (p \supset q)\ (q \supset p)$ Df.

whenever p and q are elements of K; paragraph (3) in $*1$ shows that an "asserted" proposition is at any rate a proposition; and $*4.2$ (or any other asserted formula) shows that there exists at least one element of K which is "in T." Hence, in our "arrow" notation, we have the following rules of closure on the base $(K,\ T,\ \times,\ ',\ \equiv)$:

RULES OF CLOSURE. (1). p in $K.\&.q$ in $K:\rightarrow:pq$ in K.
(2). p in $K.\rightarrow.p'$ in K.
(3). p in $K.\&.q$ in $K:\rightarrow:p\equiv q$ in K.
(4a). p in $T.\rightarrow.p$ in K.
(4b). $\exists\, p:(p$ in $T)$.

That is, our Postulates 1, 2, 3, 4a, 4b hold true in the Principia system if "$\equiv$" is replaced by "$\equiv$". [Our optional postulate 4c: $\exists\, p$ not in T, is not found in the formal Principia system; except for an "informal" convention, the formal Principia system might consist of only a single element.]

Postulate $*1.1$ is a fundamental rule of procedure which may be expressed, in our "arrow" notation, as follows:

RULE OF INFERENCE: p in $T.\&.p\supset q$ in $T:\rightarrow:q$ in T.

That is, from "p is asserted" and "$p\supset q$ is asserted" we may infer "q is asserted." This rule of procedure we shall refer to as "Inf."

The Principia's propositions $*3.2$, $*3.33$, and $*3.26$ (in view of Inf.) yield three further rules of procedure (which we shall refer to as "Adj.", "Syll.", and "Red."):

RULE OF ADJUNCTION. p in $T.\&.q$ in $T:\rightarrow:pq$ in T.
RULE OF THE SYLLOGISM. $p\supset q$ in $T.\&.q\supset r$ in $T:\rightarrow:p\supset r$ in T.
RULE OF REDUCTION. pq in $T:\rightarrow:p$ in T.

$*2.02.$ $q.\supset.p\supset q$ in T. $*2.11.$ $p\vee p'$ in T. $*2.21.$ $p'.\supset.p\supset q$ in T.
$*2.521.$ $(p\supset q)'.\supset.q\supset p$ in T. $*3.2.$ $p.\supset.(q.\supset.pq)$ in T.
$*3.24.$ $(pp')'$ in T.
$*3.26.$ $pq.\supset.p$ in T. $*3.33.$ $(p\supset q)\ (q\supset r).\supset.p\supset r$ in T.
$*3.35.$ $p(p\supset q).\supset.q$ in T.
$*4.11.p\equiv q.\equiv.p'\equiv q'$ in T. $*4.13.p\equiv p''$ in T. $*4.2.p\equiv p$ in T.
$*4.21.$ $p\equiv q.\equiv.q\equiv p$ in T. $*4.22.$ $(p\equiv q)\ (q\equiv r).\supset.p\equiv r$ in T.
$*4.24.$ $p.\equiv.pp$ in T.
$*4.3.$ $pq.\equiv.qp$ in T. $*4.32.$ $(pq)r.\equiv.p(qr)$ in T.
$*4.36.$ $p\equiv q.\supset.(pr.\equiv.qr)$ in T.
$*4.42.$ $p.\equiv.(pq.\vee.pq')$ in T. $*4.53.$ $(pq')'.\equiv.p'\vee q$ in T.
$*4.57.$ $(p'q')'.\equiv.p\vee q$ in T.
$*4.71.$ $p\supset q.\equiv.(p.\equiv.pq)$ in T. $*4.86.$ $p\equiv q.\supset.(p\equiv r.\equiv.q\equiv r)$ in T.
$*5.21.$ $p'q'.\supset.p\equiv q$ in T.

Hence our Postulates 5, 6 also hold true in the Principia system.

In the following list, formulas numbered in italics are not explicitly mentioned in the Principia, but are readily deduced by Principia methods from the Principia's postulates, and hence may properly be added to the Principia list.

$*4.03$. $(p \equiv q . \supset . p \supset q)$ in T. (By $*4.01$, $*3.26$.)
$*4.46$. $[p(p \equiv q) . \supset . q]$ in T. (By $*4.01$, $*4.32$, $*3.26$, $*4.03$, $*3.35$, Syll.)
$*4.36$. $(p \equiv q . \supset . pr \equiv qr)$ in T.
$*4.16$. $(p \equiv q . \supset . p' \equiv q')$ in T. (By $*4.11$, $*4.03$.)
$*4.86$. $(p \equiv q : \supset : p \equiv r . \equiv . q \equiv r)$ in T.
$*4.21$. $(p \equiv q . \equiv . q \equiv p)$ in T.
$*4.3$. $(pq . \equiv . qp)$ in T.
$*4.32$. $[(pq)r . \equiv . p(qr)]$ in T.
$*4.92$. $(p \equiv q : \supset : rp \equiv rq)$ in T. (By $*4.3$, $*4.46$, $*4.86$, $*4.21$, $*4.03$, $*4.36$, Syll.; cf. our proof of 24.)
$*4.93$. $(p \equiv q : \supset : r \equiv p . \equiv . r \equiv q)$ in T. (By $*4.21$, $*4.86$, $*4.03$, Syll.; cf. our proof of 25.)

Hence our Postulates 7–13, with "$\equiv$" replaced by "$\equiv$", and also our theorems 24, 25, hold true in Principia. Also,

$*4.2$. $(p \equiv p)$ in T.
$*4.26$. $(p \equiv q . \supset . q \equiv p)$ in T. (By $*4.21$, $*4.03$.)
$*4.22$. $(p \equiv q . q \equiv r : \supset : p \equiv r)$ in T.

From $*4.36$, $*4.92$, $*4.86$, $*4.93$, and $*4.16$ we derive (as in our proof of 29) the following further rule of procedure:

Rule of Replacement for $\equiv$. Let ξ be any expression built up out of parentheses, the letters $p, q, r, \ldots$, and the signs $\times, ', \equiv$ in accordance with the closure postulates, so that ξ stands for an element of K; and suppose that at some point in the expression ξ there occurs the letter p. Then if η is another expression which differs from ξ only in having this particular p replaced by q, we have the rule:

$(p \equiv q)$ in $T : \rightarrow : (\xi \equiv \eta)$ in T.

This rule (to be referred to as "Repl."), together with $*4.2$, $*4.26$, $*4.22$, shows that our theorems 26–29 (with "$\equiv$" replaced by "$\equiv$") hold true in the Principia system. Hence, *if we define "$p = q$" as an abbreviation for "$(p \equiv q)$ in T," the sign "$=$" may be handled, within this system, exactly like the ordinary equality sign in mathematics.*

In view of this fact, the following formulas serve as definitions of $\vee$ and $\supset$ in terms of $\times$, $'$, and $\equiv$.

$*4.571$. $[p \vee q. \equiv .(p'q')']$ in T. (By $*4.57$, $*4.21$.)
$*4.58$. $[p \supset q. \equiv .(pq')']$ in T. (By $*1.01$, $*4.53$.)

Hence, the Principia system may be regarded as built on the base $(K, T, \times, ', \equiv)$.

We also have:

$*4.59$. $[(p'q)'(p'q')'. \equiv .p]$ in T. (By $*4.42$, $*4.26$, $*4.571$, $*4.22$, $*4.11$.)
$*4.13$. $(p \equiv p'')$ in T.
$*5.26$. $(pp' \equiv qq')$ in T. (By $*3.24$, $*5.21$.)
$*4.24$. $(p \equiv pp)$ in T.

Hence our Postulate 14, and the theorems 31, 33, and 36, with "$\equiv$" replaced by "$=$", hold true in Principia. In view of $*5.26$ and Repl., we are justified in introducing the letter Z through the following definition:

DEFINITION. $(Z. \equiv .qq')$ in T.

The chief properties of Z (or qq') are the following:

$*5.27$. $[p(qq')'. \equiv .p]$ in T. (By $*4.59$, $*5.26$, $*4.3$, $*4.24$, $*4.13$.)
$*5.28$. $[p(qq'). \equiv .(qq')]$ in T. (By $*4.24$, $*4.26$, $*4.36$, $*4.32$, $*4.22$, $*5.26$.)

Hence our theorems 35 and 37 (with $\equiv$ replaced by $=$) hold true in Principia.

Furthermore, we are entirely at liberty to add the following definitions of p^* and $p \prec q$ to the Principia system, although we shall see in a moment that these definitions are superfluous:

DEFINITION. $[p^*. = .p \equiv (qq')]$ in T; or, briefly, $p^* = (p \equiv Z)$.
DEFINITION. $[p \prec q. = .(pq')^*]$ in T; or, briefly, $p \prec q = (pq' \equiv Z)$.

If these definitions are adopted, we find:

$*5.76$. $[(p \prec q)(q \prec p). = .(p \equiv q)]$ in T. (By $*5.38$, below, and $*4.01$.)
$*5.77$. $[(p \prec q)(q \prec r). \prec .(p \prec r)]$ in T. (By $*5.38$, below, and $*3.33$.)
$*5.78$. $[p(p \prec q) \prec q]$ in T. (By $*5.38$, below, and $*3.35$.)

Hence our Postulates 15, 16, 17 (with $\equiv$ replaced by $=$) may be said to hold true in the Principia system.

The foregoing considerations may be summed up as follows:

13.1. *Regarding the Principia system as an abstract system on the base* $(K, T, \times, ', \equiv)$, *and our system as an abstract system on the base* $(K, T, \times, ', \equiv)$, *we see that all our postulates 1–17 (except 4c), including all our definitions A–D, become valid theorems in the Principia system when* $\equiv$ *is replaced by* $\equiv$. *That is, our entire system (with* $\equiv$ *replaced by* $\equiv$*) is a formal part of the Principia system, strictly deducible from the Principia's postulates.*

The converse, however, is not true. There is one postulate of the Principia system which does not correspond to anything in our system, namely, ∗4.01, which may be written as follows:

∗*4.04*. $[(p \supset q)(q \supset p).\equiv.(p \equiv q)]$ in T. (By ∗4.01, ∗4.2.)

The following important theorems are consequences of ∗*4.04*.

∗*5.37*. $[p.\equiv.p \equiv (qq')']$ in T. (By ∗4.01, ∗*4.58*, ∗4.13, ∗*5.28*, ∗*5.27*.)

∗*5.38*. $[p'.\equiv.p \equiv (qq')]$ in T. (By ∗4.01, ∗*4.58*, ∗*5.27*, ∗*5.28*.)

Or briefly (using the notation $p = q$ for $p \equiv q$ in T, and Z for qq'):

$$p = (p \equiv Z') \qquad \text{and} \qquad p' = (p \equiv Z).$$

We now see why the definitions of p^* and $p \prec q$ are superfluous in the Principia system; for, in view of ∗*5.38* and ∗*4.58*, these definitions become

$$p^* = p' \qquad \text{and} \qquad p \prec q.=.p \supset q.$$

We also observe that it is theorem ∗*5.38* $[p' = (p \equiv Z)]$ which is responsible for the so-called paradoxes of the Principia system, such as the following:

P1. q in $T.\rightarrow.(p \supset q)$ in T. (By ∗2.02.)
P2. p' in $T.\rightarrow.(p \supset q)$ in T. (By ∗2.21.)
P3. $(p \supset q)'$ in $T.\rightarrow.(q \supset p)$ in T. (By ∗2.521.)

Also, using the notation p in T' for p' in T (cf. our 53), and the notation $p = Z$ for $p \equiv q$ in T, we have

P4. p in $T.\rightarrow.p = Z'$. (By ∗*5.37*.)
P5. p in $T'.\rightarrow.p = Z$. (By ∗*5.38*.)

where $Z = qq'$. That is, all the elements in T are interchangeable with the unique element Z', and all the elements in T' are interchangeable with the unique element Z. It should be noted, however, that there is nothing in the formal Principia system to exclude elements which are neither in T nor in T' (see our Example B9, above). [It is only an "informal" reading of the formula ∗2.11: $p \vee p'$ in T which

makes it appear that every element of a Principia system must be either in T or in T'.]

The essential contrast between the two systems may therefore be summarized as follows:

13.2. *The Principia system* $(K, T, \times, ', \equiv)$ *and our system* $(K, T, \times, ', \equiv)$ *are mathematically identical (that is, isomorphic), when* $\equiv$ *is replaced by* $=$, *except that the Principia system includes one additional postulate, which may be stated briefly in the form* $p' = (p \equiv Z)$.

In short, the Principia system identifies the two elements p' and $p \equiv Z$, while our system (following Lewis) permits the two elements p' and $p \equiv Z$ to be treated as separate entities.

Hence, the concept of "material" implication, defined by $p \supset q. = .pq' \equiv Z$, or $p \supset q. = .pq \equiv p$ ($*4.71$), is less flexible (and for many purposes less useful) than the concept of "strict" implication, defined by $p \prec q. = .pq' \equiv Z$, or $p \prec q. = .pq \equiv p$ (our 74).

HARVARD UNIVERSITY,
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